

# A differential version of the Chebyshev-Markov-Stieltjes inequalities

Shoni Gilboa\*

Ron Peled†

February 10, 2015

## Abstract

We show that a differential version of the classical Chebyshev-Markov-Stieltjes inequalities holds for a broad family of weight functions. Such a differential version appears to be new. Our results apply to weight functions which are bounded away from zero and piecewise absolutely continuous and yield effective estimates when the weight satisfies additional regularity conditions.

## 1 Introduction

Let  $w$  be a non-negative, integrable function on the interval  $[-1, 1]$ , with non-zero integral. Let  $n \geq 1$  and let  $\mathcal{M}_n$  be the class of all positive Borel measures  $\mu$  on  $[-1, 1]$  satisfying that

$$\int_{-1}^1 p(t)w(t)dt = \int p(t)d\mu(t)$$

for all polynomials  $p$  of degree at most  $n$ . Define the extremal functions

$$\pi(x) := \sup_{\mu \in \mathcal{M}_n} \mu([-1, x]), \quad \underline{\pi}(x) := \inf_{\mu \in \mathcal{M}_n} \mu([-1, x]), \quad \lambda(x) := \sup_{\mu \in \mathcal{M}_n} \mu(\{x\}). \quad (1)$$

Figures 1 and 2 depict these functions for a specific choice of weight function. Trivially,

$$\underline{\pi}(x) \leq \int_{-1}^x w(t)dt \leq \pi(x), \quad -1 \leq x \leq 1. \quad (2)$$

Investigations going back to the work of Chebyshev, Markov and Stieltjes have shown that for each  $-1 \leq x \leq 1$  there is a unique measure simultaneously attaining the suprema and infimum in (1). These extremal measures form an explicit one-parameter family of atomic measures in  $\mathcal{M}_n$ , termed the *canonical representations*. In Section 2 we review the theory of canonical representations. The explicit identification of the extremal measures makes the inequalities (2) into a powerful tool, termed

---

\*Mathematics Dept., The Open University of Israel, Raanana 43107, Israel, Email: tipshoni@gmail.com.

†Mathematics Dept., Tel-Aviv University, Tel-Aviv 69978, Israel, Email: peledron@post.tau.ac.il. Supported by an ISF grant and an IRG grant.

the Chebyshev-Markov-Stieltjes inequalities. One consequence of the fact that the extrema in (1) are attained on the same measure is the following relation between the extremal functions,

$$\pi(x) - \underline{\pi}(x) = \lambda(x).$$

This implies, together with (2), that

$$\left| \pi(x) - \int_{-1}^x w(t) dt \right| \leq \lambda(x) \quad \text{and} \quad \left| \underline{\pi}(x) - \int_{-1}^x w(t) dt \right| \leq \lambda(x), \quad -1 \leq x \leq 1. \quad (3)$$

In this paper we show that the inequalities (3) may be valid also in a pointwise rather than cumulative sense. We focus on the class of weight functions which are bounded away from zero and satisfy certain regularity conditions.

Observe that  $\pi$  is a non-decreasing function and thus differentiable almost everywhere on  $(-1, 1)$ . In fact, more is true, the function  $\pi$  is analytic at all but finitely many points of  $(-1, 1)$ , where the exceptional points are roots of the principal representations, as elaborated in Section 2.

**Theorem 1.1.** *Suppose  $w$  is a function on  $[-1, 1]$  satisfying that for some  $m, R > 0$ :*

- $w(x) \geq m$  for every  $-1 \leq x \leq 1$ .
- $|w(x) - w(y)| \leq R \cdot (y - x)$  for every  $-1 \leq x < y \leq 1$ .

*Then there exists a constant  $C(w) > 0$  such that for every differentiability point  $x \in (-1, 1)$  of  $\pi$ ,*

$$-C(w) \frac{\lambda(x)}{1-x} \leq \pi'(x) - w(x) \leq C(w) \lambda(x) \min \left\{ \frac{1}{1+x}, n^2 \right\}. \quad (4)$$

This inequality appears to be new. One motivation for its development came from papers of Kuijlaars [8, 9] where a lower bound for  $\pi'$  for the case of Jacobi weight functions played a central role. Figure 3 presents a plot of the function  $\pi' - w$ . The constant  $C(w)$  appearing in the inequality may be given an explicit estimate depending only on  $m$  and  $R$ , yielding uniformity of our estimates for the class of weight functions satisfying the assumptions of the theorem, see the discussion in Section 8.

It is worth noting that when  $w$  is bounded above, the function  $\lambda(x)$  cannot be too large. Specifically, it satisfies

$$\lambda(x) \leq \frac{CM}{n} \max \left\{ \sqrt{1-x^2}, \frac{1}{n} \right\}, \quad -1 < x < 1,$$

where  $C > 0$  is an absolute constant and  $M$  is the maximum of  $w$  on  $[-1, 1]$  (see Lemma 3.8 below). Thus the above theorem may also be seen as a practical tool for estimating the density  $w$  if one is given only the first few moments of the measure  $w(t)dt$ . With regards to this we mention that Lemma 6.1 gives an explicit expression for  $\pi'$ .

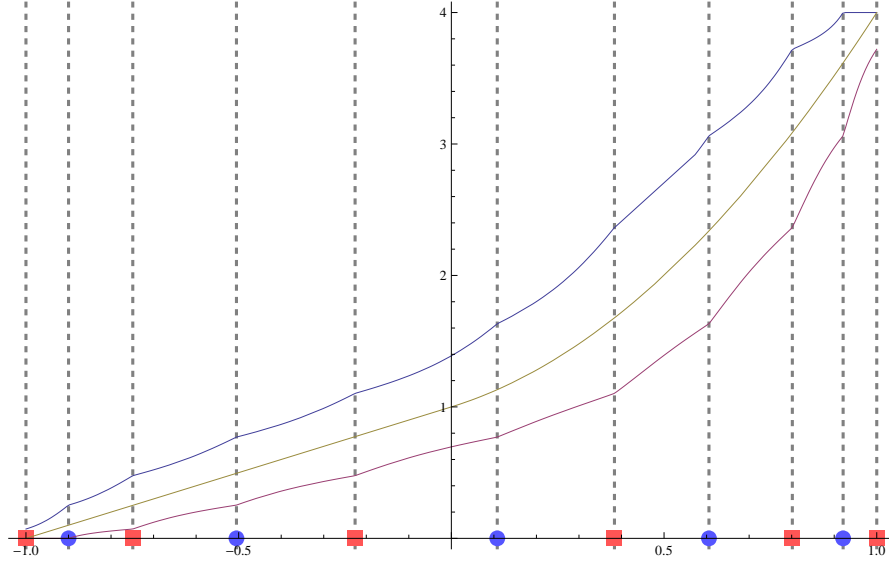


Figure 1: A plot of  $\pi$  (top graph),  $\int_{-1}^x w(t)dt$  (middle graph) and  $\underline{\pi}$  (bottom graph) for  $n = 5$  and the weight function  $w(t) = \max\{1, 1 + 4t\}$ . The circles on the axis denote the nodes of the Gaussian quadrature and the squares denote the nodes of the Lobatto quadrature. These quadratures are defined in Section 2, where the fact that  $\pi$  is constant to the right of the last Gaussian node is also explained.

We also remark that in Theorem 1.1, as well as the next two theorems, one immediately obtains corresponding bounds with  $\underline{\pi}$  replacing  $\pi$  by considering the reversed weight function  $\tilde{w}(x) := w(-x)$ , as we have the identity  $\pi^{\tilde{w}}(x) = \int_{-1}^1 w(t)dt - \underline{\pi}^w(-x)$ .

The next two theorems provide estimates for the difference  $\pi' - w$  under weaker regularity assumptions on  $w$ . In the first of these the assumption of Lipschitz continuity is relaxed to a Sobolev condition. In the second, the regularity assumptions on  $w$  are relaxed to mere piecewise absolute continuity, thus allowing a class of discontinuous weight functions, at the price of making the result non-quantitative.

**Theorem 1.2.** *Suppose  $w$  is a function on  $[-1, 1]$  satisfying that for some  $m > 0$  and  $p > 1$ :*

- $w(x) \geq m$  for every  $-1 \leq x \leq 1$ .
- $w$  is absolutely continuous and  $w' \in L_p[-1, 1]$ .

*Then there exists a constant  $C(w) > 0$  such that for every differentiability point  $x \in (-1, 1)$  of  $\pi$ ,*

$$-C(w) \left( \frac{1}{1-x} + \frac{1}{\lambda(x)^{1/p}} \right) \lambda(x) \leq \pi'(x) - w(x) \leq C(w) \left( \min \left\{ \frac{1}{1+x}, n^2 \right\} + \frac{1}{\lambda(x)^{1/p}} \right) \lambda(x). \quad (5)$$

**Theorem 1.3.** *Suppose  $w$  is a function on  $[-1, 1]$  satisfying that for some  $m > 0$  and  $-1 < s_1 < s_2 < \dots < s_L < 1$ :*

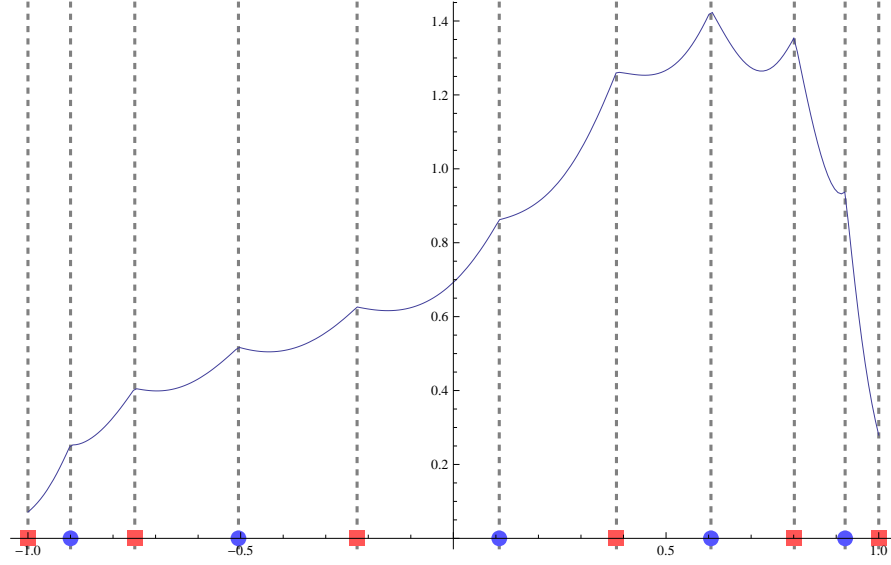


Figure 2: A plot of the function  $\lambda$  for  $n = 5$  and the weight function  $w(t) = \max\{1, 1 + 4t\}$ . The circles on the axis denote the nodes of the Gaussian quadrature and the squares denote the nodes of the Lobatto quadrature as defined in Section 2.

- $w(x) \geq m$  for every  $-1 \leq x \leq 1$ .
- $w$  is absolutely continuous on each of the intervals  $[-1, s_1), (s_1, s_2), \dots, (s_L, 1]$ .

Then for every  $\varepsilon > 0$  there exists a  $C(w, \varepsilon) > 0$  such that

$$|\pi'(x) - w(x)| \leq \varepsilon \quad (6)$$

for every differentiability point  $x \in (-1, 1)$  of  $\pi$  satisfying  $1 - x^2 \geq \frac{C(w, \varepsilon)}{n^2}$  and  $|s_i - x| \geq \frac{C(w, \varepsilon)}{n}$  for  $1 \leq i \leq L$ .

**Remark.** The absolute continuity assumption in Theorem 1.3 amounts to saying that  $w$  is the sum of an absolutely continuous function on  $[-1, 1]$  and a step function with discontinuities only at the points  $s_1, s_2, \dots, s_L$ . In particular,  $w$  has left and right limits at each of the  $s_i$ .

A more refined version of Theorem 1.3 may be obtained using Remark 7.4.

Section 2 presents the notation and background used throughout the paper. In Section 3 we gather several estimates for orthogonal polynomials and quadrature formulas which will be used throughout the proofs. The estimates of this section are all essentially known but the concise presentation may be of use to a non-expert in the literature. Accordingly, we include short proofs for most of the statements there, relying on various comparison arguments to the case of the constant weight function. An important role in these is played by a theorem of Badkov, see Theorem 3.1 below. In Section 4 we provide lower bounds on the distance between nodes of canonical representations. In

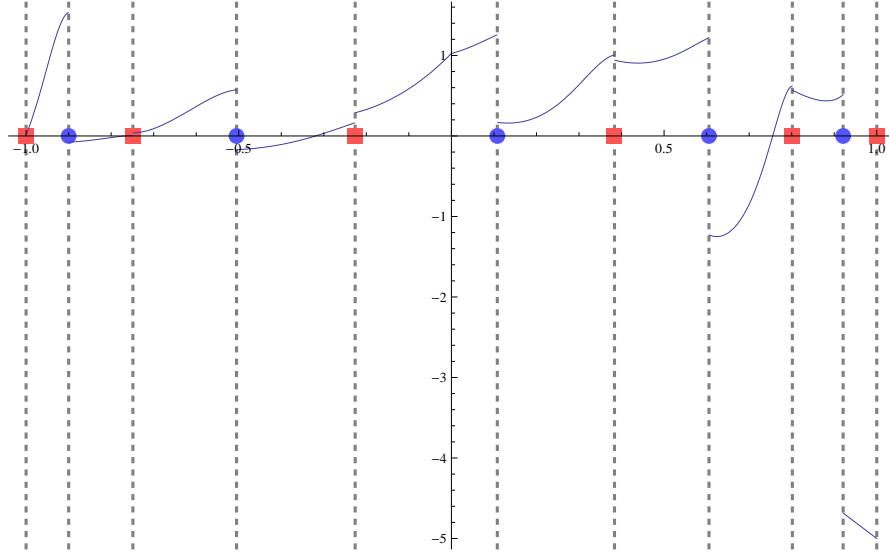


Figure 3: A plot of the function  $\pi' - w$  for  $n = 5$  and the weight function  $w(t) = \max\{1, 1 + 4t\}$ . The circles on the axis denote the nodes of the Gaussian quadrature and the squares denote the nodes of the Lobatto quadrature as defined in Section 2. The lack of smoothness at  $t = 0$  is due to the lack of smoothness of  $w$  at this point as we know that  $\pi$  is analytic there, see Section 2.

In Section 5 we consider the polynomials whose roots are the nodes of the canonical representations and establish lower bounds for their derivatives at these nodes. We also prove there that the interpolation polynomials corresponding to the canonical representations exhibit strong localization properties. Theorems 1.1 and 1.2 are proved in Section 6. Theorem 1.3 is proved in Section 7. In the final Section 8 we discuss some open questions.

## 2 Notation and background

Throughout the paper we use the following notation. We refer the reader to the book of Karlin and Studden [7], especially to section 2 in chapter IV there, for reference to the facts mentioned in this section.

Let  $w$  be a non-negative, integrable function on the interval  $[-1, 1]$ , with non-zero integral. Such a  $w$  will be called a *weight function*. Let  $n$  be a positive integer and let  $\varphi$  be the  $n$ th-degree orthonormal polynomial, with positive leading coefficient, with respect to  $w$ . Let  $\psi$  be the  $(n - 1)$ th-degree orthonormal polynomial, with positive leading coefficient, with respect to  $(1 - t^2)w(t)$ . Figure 4 depicts these polynomials for  $n = 5$  and the weight function  $w(t) = \max\{1, 1 + 4t\}$ .

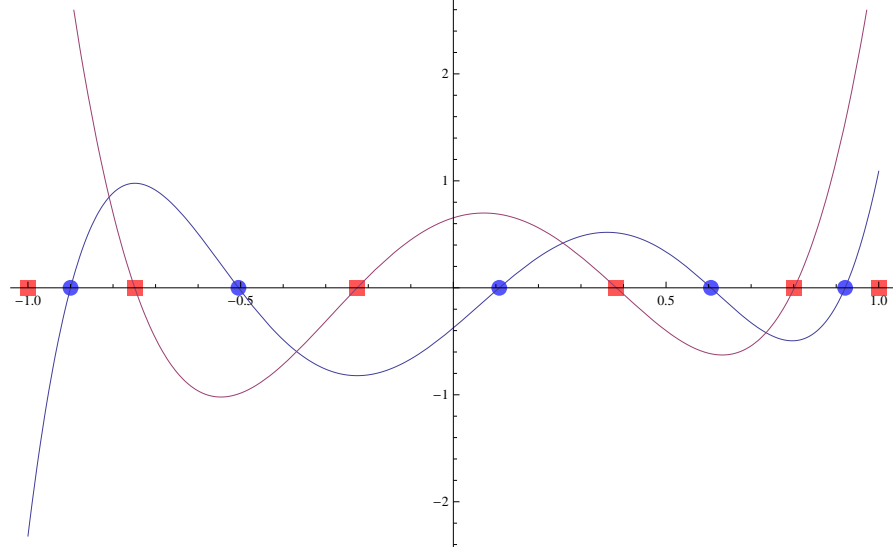


Figure 4: A plot of the polynomials  $\varphi$  and  $\psi$  for  $n = 5$  and the weight function  $w(t) = \max\{1, 1 + 4t\}$ . The circles on the axis denote the nodes of the Gaussian quadrature and the squares denote the nodes of the Lobatto quadrature.

For every real  $a$  define the polynomial

$$P_a(t) := \begin{cases} \varphi(t) - a(1-t)\psi(t) & \text{if } a \geq 0 \\ \varphi(t) - a(1+t)\psi(t) & \text{if } a \leq 0 \end{cases}. \quad (7)$$

$P_a$  has  $n$  simple zeros in  $(-1, 1)$  which we denote by

$$-1 < \xi_1(a) < \dots < \xi_n(a) < 1.$$

Also  $(1 - t^2)\psi(t)$  has  $n + 1$  simple roots which we denote by

$$-1 = \eta_0 < \eta_1 < \dots < \eta_n = 1.$$

We will make use of the fact that

$$\text{as } a \text{ increases from } -\infty \text{ to } \infty, \xi_i(a) \text{ increases from } \eta_{i-1} \text{ to } \eta_i. \quad (8)$$

Consistently with this fact, we denote  $\xi_i(-\infty) := \eta_{i-1}$ ,  $\xi_i(\infty) := \eta_i$ .

A *quadrature* (formula) of degree  $k$  for the weight function  $w$  is a set of points  $-1 \leq t_1 \leq t_2 \leq \dots \leq t_N \leq 1$ , called *nodes*, and a set of *weights*  $w_1, w_2, \dots, w_N$  such that

$$\int_{-1}^1 p(t)w(t)dt = \sum_{i=1}^N w_i p(t_i) \quad (9)$$

for every polynomial  $p$  of degree at most  $k$ . Define the *index* function

$$I(u) := \begin{cases} 2 & -1 < u < 1 \\ 1 & u \in \{-1, 1\} \end{cases}. \quad (10)$$

Define also the index of a quadrature formula to be the sum of the indices of its nodes.

We now describe all the degree  $2n - 1$  quadrature formulas for  $w$ , having positive weights, whose index is at most  $2n + 1$ . These formulas are called *canonical representations*. We describe the set of nodes of these formulas, considering separately 4 cases:

1. The roots  $(\xi_i(0))$  are the nodes of a quadrature formula, known as the *Gaussian quadrature* or the lower principal representation (see also Figure 4).
2. The roots  $(\eta_i)$  are the nodes of a quadrature formula, known as the *Lobatto quadrature* or the upper principal representation (see also Figure 4).
3. For every  $0 < a < \infty$  the roots  $(\xi_i(a))$  with the additional node  $-1$  are the nodes of a quadrature formula, called a lower canonical representation.
4. For every  $-\infty < a < 0$  the roots  $(\xi_i(a))$  with the additional node  $1$  are the nodes of a quadrature formula, called an upper canonical representation.

Every  $-1 < x < 1$  is therefore a node of exactly one of these quadrature formulas, and we denote this quadrature formula by  $\Sigma_x$  and its set of nodes by  $S_x$ .

It is a classical fact that for every  $-1 < x < 1$ , the suprema and infimum in (1) are simultaneously attained on the quadrature formula  $\Sigma_x$  (and for  $x \in \{-1, 1\}$  they are attained on the Lobatto quadrature). Correspondingly, for  $-1 < x < 1$  and  $-1 \leq u \leq 1$  we denote by  $\lambda_x(u)$  the weight of  $u$  in  $\Sigma_x$  and, for brevity, we write  $\lambda(x)$  instead of  $\lambda_x(x)$ . Thus,  $\lambda_x(u) \neq 0$  if and only if  $u \in S_x$ , and  $\lambda_x(u) = \lambda(u)$  if  $u \in S_x - \{-1, 1\}$ . Figure 2 shows a plot of  $\lambda$  for a certain choice of weight function. In addition, we have that

$$\pi(x) = \sum_{u \in S_x \cap [-1, x]} \lambda_x(u) \quad \text{and} \quad \underline{\pi}(x) = \sum_{u \in S_x \cap [-1, x)} \lambda_x(u) \quad (11)$$

are the weights given to the intervals  $[-1, x]$  and  $[-1, x)$ , respectively, by the quadrature formula  $\Sigma_x$ . Figure 1 shows a plot of  $\pi$  and  $\underline{\pi}$  for a certain choice of weight function. It follows that for  $x > \xi_n(0)$  we have  $\pi(x) = \int_{-1}^1 w(t)dt$ , a fact which is clearly visible in the figure.

We remark also, as mentioned in the introduction, that the function  $\pi$  (and similarly  $\underline{\pi}$ ) is analytic at all  $x \in (-1, 1)$  which are not roots of the Gaussian or the Lobatto quadratures. Indeed, fix  $1 \leq i \leq n$  and an interval  $I \subseteq \mathbb{R} - \{0\}$  and let us show that  $\pi$  is analytic on the interval  $\{x: x = \xi_i(a), a \in I\}$ . Assume that  $I \subseteq (0, \infty)$  as the case  $I \subseteq (-\infty, 0)$  is treated similarly. Let  $p_{i,a}(\cdot)$  be the minimum degree polynomial satisfying  $p_{i,a}(-1) = 1$ ,  $p_{i,a}(\xi_j(a)) = 1$  for  $1 \leq j \leq i$

and  $p_{i,a}(\xi_j(a)) = 0$  for  $i < j \leq n$ . Since  $p_{i,a}$  has degree smaller or equal to  $2n - 1$  we have  $\pi(\xi_i(a)) = \int_{-1}^1 p_{i,a}(t)w(t)dt$  for all  $a \in I$  by (11). We conclude that  $\pi(\xi_i(a))$  is a rational function of  $\xi_1(a), \dots, \xi_n(a)$  for  $a \in I$ . In addition, a simple use of the implicit function theorem shows that  $\xi_1(a), \dots, \xi_n(a)$  are analytic functions of  $a$  for  $a \in I$ . Thus we are done by observing that for  $a \in I$ ,  $a = \frac{\varphi(\xi_i(a))}{(1-\xi_i(a))\psi(\xi_i(a))}$  is a rational function of  $\xi_i(a)$ .

Throughout the paper we adopt the following policy regarding constants. The constants  $C(w)$  and  $c(w)$  will always denote positive constants whose value depends only on the function  $w$ . The values of these constants will be allowed to change from line to line, even within the same calculation, with the value of  $C(w)$  increasing and the value of  $c(w)$  decreasing. We similarly use  $C$  and  $c$  to denote positive absolute constants whose value may change from line to line.

### 3 Preliminary estimates

This section collects several estimates, all essentially known, on the polynomials  $\varphi$ ,  $\psi$  and  $P_a$ , their zeros, and the associated quadrature formulas.

#### 3.1 Orthogonal polynomials

The following theorem, which gives lower and upper bounds on the orthogonal polynomials  $\varphi$  and  $\psi$ , plays an important role in our work. The theorem is a corollary of results of Badkov [1, Theorem 1.2 and Theorem 1.4] (see also Theorem 13.2 there). The results of Badkov hold in great generality, yielding estimates for orthogonal polynomials for a broad family of weights, and are mostly stated in terms of trigonometric orthogonal polynomials. Here we state a reformulation in terms of algebraic orthogonal polynomials, of the special case that we need.

**Theorem 3.1** (Badkov). *Suppose that the weight function  $w$  has the form*

$$w(x) = h(x)(1 - x^2)^\alpha \quad \alpha > -1,$$

*where the function  $h$  is assumed to be a measurable function on  $[-1, 1]$ , satisfying  $0 < m \leq h \leq M$  almost everywhere and the ‘mean-squared Hölder condition’*

$$\forall \delta > 0, \quad \int_{-\pi}^{\pi} (h(\cos(\theta + \delta)) - h(\cos \theta))^2 d\theta \leq C\delta \quad (12)$$

*for some constants  $M, m, C > 0$ . Then there exist constants  $C(w), c(w) > 0$  such that for all  $-1 < x < 1$ ,*

$$c(w) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{\alpha+1/2} \leq |\varphi(x)| + \sqrt{1-x^2} |\psi(x)| \leq C(w) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{\alpha+1/2}. \quad (13)$$



We stress that the assumptions in the above theorem may hold even for weight functions  $w$  with discontinuities in  $(-1, 1)$ . For our purposes we shall use the theorem when  $\alpha \in \{0, 1\}$  and when  $h$  is itself a weight function satisfying the assumptions of Theorem 1.1, Theorem 1.2 or Theorem 1.3. For instance, if  $h$  is absolutely continuous we may verify condition (12) by noting that for every  $\delta > 0$ ,

$$\begin{aligned} \int_{-\pi}^{\pi} (h(\cos(\theta + \delta)) - h(\cos \theta))^2 d\theta &\leq \int_{-\pi}^{\pi} |h(\cos(\theta + \delta)) - h(\cos \theta)| d\theta \max_{-1 \leq x \leq 1} h(x) = \\ &= \int_{-\pi}^{\pi} \left| \int_{\cos \theta}^{\cos(\theta + \delta)} h'(t) dt \right| d\theta \max_{-1 \leq x \leq 1} h(x) \leq \int_{-\pi}^{\pi} \int_{\min(\cos \theta, \cos(\theta + \delta))}^{\max(\cos \theta, \cos(\theta + \delta))} |h'(t)| dt d\theta \max_{-1 \leq x \leq 1} h(x) \leq \\ &\leq \left( 2 \int_{-1}^1 |h'(t)| dt \max_{-1 \leq x \leq 1} h(x) \right) \delta, \end{aligned}$$

where in the last inequality we have changed the order of integration and used the fact that for each  $-1 \leq t \leq 1$ , the set of  $\theta$  for which  $t$  is between  $\cos(\theta)$  and  $\cos(\theta + \delta)$ , when viewed on the circle, is contained in the union of the segments  $[\arccos(t) - \delta, \arccos(t)]$  and  $[2\pi - \arccos(t) - \delta, 2\pi - \arccos(t)]$ .

The above theorem seems to be the most complicated property of orthogonal polynomials which we require. When the function  $h$  satisfies the assumptions of Theorem 1.1 the proof of the theorem is simpler, making use of the Korovs comparison theorem [10, Theorem 7.1.3]. For completeness, the proof in this case and an overview of the general case are provided in the Appendix.

The following corollary summarizes the bounds on orthogonal polynomials which we will need for this work.

**Corollary 3.2.** *Suppose  $w$  satisfies the conditions of Theorem 3.1 with  $\alpha = 0$ . Then there exist constants  $C(w), c(w) > 0$  such that for every  $-1 < x < 1$ ,*

$$|\varphi(x)| \leq C(w) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{1/2}, \quad (14)$$

$$|\psi(x)| \leq C(w) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{3/2}, \quad (15)$$

$$|\varphi(x)| + (1-x^2) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\} |\psi(x)| \geq c(w) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{1/2}. \quad (16)$$

*Proof.* The corollary follows directly from Theorem 3.1. The bound (14) follows from the case  $\alpha = 0$ . The bound (15) follows from the case  $\alpha = 1$  using the upper bound on  $\varphi$  in (13). For the bound (16), first observe that by the case  $\alpha = 0$ ,

$$|\varphi(x)| + \sqrt{1-x^2} |\psi(x)| \geq c(w) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{1/2}, \quad -1 < x < 1. \quad (17)$$

Together with the fact that for every  $0 < \varepsilon < 1$  we have

$$|\varphi(x)| + (1-x^2) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\} |\psi(x)| \geq \sqrt{\varepsilon} \left( |\varphi(x)| + \sqrt{1-x^2} |\psi(x)| \right), \quad 1-x^2 \geq \varepsilon n^{-2},$$

we see that we need only prove (16) when  $1 - x^2 < \varepsilon n^{-2}$  for some fixed  $0 < \varepsilon < 1$ . To this end, observe that by (15) we have

$$|\psi(x)| \leq C(w)n^{3/2}, \quad -1 < x < 1.$$

Substituting this relation into (17) yields that for every  $0 < \varepsilon < 1$ ,

$$|\varphi(x)| \geq (c(w) - \sqrt{\varepsilon}C(w))n^{1/2}, \quad 1 - x^2 \leq \varepsilon n^{-2}.$$

Fixing  $\varepsilon$  sufficiently small so that the right-hand side is positive proves (16) when  $1 - x^2 < \varepsilon n^{-2}$ .  $\square$

The following proposition uses the upper bounds of the previous corollary to obtain derivative bounds. To obtain unified expressions we make use of the sign function, defined by

$$\text{sgn}(a) := \begin{cases} 1 & a > 0 \\ 0 & a = 0 \\ -1 & a < 0. \end{cases} \quad (18)$$

**Proposition 3.3.** *Suppose  $w$  satisfies the conditions of Theorem 3.1 with  $\alpha = 0$ . Then there exists a constant  $C(w) > 0$  such that for every  $-1 < x < 1$ ,*

$$|\varphi'(x)| \leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{3/2}, \quad (19)$$

$$|\psi'(x)| \leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{5/2}, \quad (20)$$

$$|P'_a(x)| \leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{3/2} \left( 1 + |a| \left( (1 - \text{sgn}(a)x) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\} + \frac{1}{n} \right) \right). \quad (21)$$

*Proof.* We use a combination of the Bernstein (see [10, Theorem 1.22.3]) and A. Markov (see [3, Theorem 5.1.8]) inequalities, which states that for any polynomial  $p$  and every  $-\alpha < x < \alpha$ ,

$$|p'(x)| \leq \deg p \cdot \min \left\{ \frac{1}{\sqrt{\alpha^2 - x^2}}, \frac{\deg p}{\alpha} \right\} \max_{-\alpha \leq t \leq \alpha} |p(t)|. \quad (22)$$

Fix  $-1 < x < 1$  and denote  $\rho := \sqrt{1 - x^2}$ . First we prove (19). Let  $I := \left[ -\sqrt{1 - \frac{1}{2}\rho^2}, \sqrt{1 - \frac{1}{2}\rho^2} \right]$ . Observe that  $x \in I$ ,  $|I| \geq \sqrt{2}$  and  $1 - t^2 \geq \rho^2/2$  when  $t \in I$ . By (14),

$$\max_{t \in I} |\varphi(t)| \leq \max_{t \in I} C(w) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^{1/2} \leq C(w) \min \left\{ n, \frac{\sqrt{2}}{\rho} \right\}^{1/2}.$$

Thus, using (22) for the interval  $I$  yields

$$|\varphi'(x)| \leq n \cdot \min \left\{ \frac{\sqrt{2}}{\rho}, \frac{2n}{|I|} \right\} C(w) \min \left\{ n, \frac{\sqrt{2}}{\rho} \right\}^{1/2} \leq C(w)n \cdot \min \left\{ n, \frac{1}{\rho} \right\}^{\frac{3}{2}}.$$

The bound (20) follows similarly. To get (21), note that

$$P'_a(x) = \begin{cases} \varphi'(x) - a(1-x)\psi'(x) + a\psi(x) & a > 0 \\ \varphi'(x) - a(1+x)\psi'(x) - a\psi(x) & a < 0 \end{cases}$$

and use (15), (19) and (20).  $\square$

We remark that certain lower bounds on the derivative of  $P_a$  and  $\psi$  are proved in Section 5.

## 3.2 Quadrature formulas

In this section we will sometimes have need to consider two weight functions simultaneously. In such cases, to avoid ambiguity, we add the superscript  $w$  to quantities such as  $\xi, \eta, \lambda$  and  $\pi$  to indicate that the weight function is  $w$ .

### 3.2.1 Distance of nodes from endpoints

Let  $S$  be the set of all non-zero polynomials  $p$  satisfying  $\deg(p) \leq 2n - 2$  and  $p \geq 0$  on  $[-1, 1]$ . The following lemma gives a max-min formula for  $\xi_i$  involving polynomials in  $S$ .

**Lemma 3.4.** *For every weight function  $w$ ,*

$$\xi_i(0) = \max_{-1 < z_1 < \dots < z_{i-1} < 1} \min_{\substack{p \in S \\ p(z_1) = \dots = p(z_{i-1}) = 0}} \frac{\int_{-1}^1 tp(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt}, \quad 1 \leq i \leq n. \quad (23)$$

*Proof.* For brevity, we denote in this proof the zeros of  $\varphi$  by  $\xi_i$  instead of  $\xi_i(0)$ .

Recall from Section 2 the notation  $\Sigma_0$  for the quadrature formula of degree  $2n - 1$  whose nodes are  $(\xi_i)$ . Using this quadrature formula to evaluate the integrals we obtain for every  $p \in S$  that

$$\frac{\int_{-1}^1 tp(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt} = \frac{\sum_{j=1}^n \lambda(\xi_j) \xi_j p(\xi_j)}{\sum_{j=1}^n \lambda(\xi_j) p(\xi_j)}. \quad (24)$$

Fix  $1 \leq i \leq n$ . We prove (23) by establishing that the right-hand side is both an upper and lower bound for  $\xi_i$ . First, let  $-1 < z_1 < \dots < z_{i-1} < 1$  and consider the polynomial

$$q(t) := \prod_{j=1}^{i-1} (t - z_j)^2 \cdot \prod_{j=i+1}^n (t - \xi_j)^2.$$

This polynomial belongs to  $S$  and vanishes at  $z_1, \dots, z_{i-1}$ . Thus, using (24),

$$\min_{\substack{p \in S \\ p(z_1) = \dots = p(z_{i-1}) = 0}} \frac{\int_{-1}^1 tp(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt} = \min_{\substack{p \in S \\ p(z_1) = \dots = p(z_{i-1}) = 0}} \frac{\sum_{j=1}^n \lambda(\xi_j) \xi_j p(\xi_j)}{\sum_{j=1}^n \lambda(\xi_j) p(\xi_j)} \leq \frac{\sum_{j=1}^n \lambda(\xi_j) \xi_j q(\xi_j)}{\sum_{j=1}^n \lambda(\xi_j) q(\xi_j)}.$$

Since  $q$  vanishes at  $\xi_{i+1}, \dots, \xi_n$  and  $\xi_1 < \dots < \xi_i$ , it follows that

$$\min_{\substack{p \in S \\ p(z_1)=\dots=p(z_{i-1})=0}} \frac{\int_{-1}^1 tp(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt} \leq \frac{\sum_{j=1}^i \lambda(\xi_j)\xi_j q(\xi_j)}{\sum_{j=1}^i \lambda(\xi_j)q(\xi_j)} \leq \xi_i.$$

Since the  $(z_j)$  are arbitrary, we conclude that

$$\xi_i \geq \max_{-1 < z_1 < \dots < z_{i-1} < 1} \min_{\substack{p \in S \\ p(z_1)=\dots=p(z_{i-1})=0}} \frac{\int_{-1}^1 tp(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt}. \quad (25)$$

Second, suppose  $p$  is a polynomial in  $S$  which vanishes at  $z_1 = \xi_1, \dots, z_{i-1} = \xi_{i-1}$ . It follows from (24) that

$$\frac{\int_{-1}^1 tp(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt} = \frac{\sum_{j=1}^n \lambda(\xi_j)\xi_j p(\xi_j)}{\sum_{j=1}^n \lambda(\xi_j)p(\xi_j)} = \frac{\sum_{j=i}^n \lambda(\xi_j)\xi_j p(\xi_j)}{\sum_{j=i}^n \lambda(\xi_j)p(\xi_j)} \geq \xi_i.$$

Hence,

$$\xi_i \leq \min_{\substack{p \in S \\ p(\xi_1)=\dots=p(\xi_{i-1})=0}} \frac{\int_{-1}^1 tp(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt} \leq \max_{-1 < z_1 < \dots < z_{i-1} < 1} \min_{\substack{p \in S \\ p(z_1)=\dots=p(z_{i-1})=0}} \frac{\int_{-1}^1 tp(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt}. \quad (26)$$

The lemma follows by putting together (25) and (26).  $\square$

Denote by  $u$  the constant weight function,  $u \equiv 1$  on  $[-1, 1]$ . Let

$$L_n(t) := \frac{1}{2^n n!} \frac{d^n}{dt^n} [(t^2 - 1)^n] \quad (27)$$

be the Legendre polynomial of degree  $n$ . As is well-known, the polynomials  $(L_n)$  are orthogonal with respect to  $u$ . By our notation,  $(\xi_i^u(0))$ ,  $1 \leq i \leq n$ , are the roots of  $L_n$ . We require the following bounds on these roots [10, Theorem 6.21.2],

$$-\cos\left(\frac{2i-1}{2n+1}\pi\right) \leq \xi_i^u(0) \leq -\cos\left(\frac{2i}{2n+1}\pi\right), \quad 1 \leq i \leq n. \quad (28)$$

**Corollary 3.5.** *Suppose the weight function  $w$  satisfies  $0 < m \leq w \leq M$  almost everywhere. Then for each  $1 \leq i \leq n$ ,*

$$\frac{m}{M}(1 - \xi_i^u(0)) \leq 1 - \xi_i^w(0) \leq \frac{M}{m}(1 - \xi_i^u(0)) \quad \text{and} \quad \frac{m}{M}(1 + \xi_i^u(0)) \leq 1 + \xi_i^w(0) \leq \frac{M}{m}(1 + \xi_i^u(0)). \quad (29)$$

Consequently, there exist absolute constants  $C, c > 0$  such that for all  $1 \leq i \leq n$ ,

$$\begin{aligned} c\sqrt{\frac{m}{M}} \cdot \frac{n+1-i}{n} &\leq \sqrt{1 - \xi_i^w(0)} \leq C\sqrt{\frac{M}{m}} \cdot \frac{n+1-i}{n} \quad \text{and} \\ c\sqrt{\frac{m}{M}} \cdot \frac{i}{n} &\leq \sqrt{1 + \xi_i^w(0)} \leq C\sqrt{\frac{M}{m}} \cdot \frac{i}{n}. \end{aligned} \quad (30)$$

*Proof.* Applying Lemma 3.4 twice, once for  $w$  and once for  $u$ , we obtain

$$\begin{aligned} 1 - \xi_i^w(0) &= \min_{-1 < z_1 < \dots < z_{i-1} < 1} \max_{\substack{p \in S \\ p(z_1) = \dots = p(z_{i-1}) = 0}} \frac{\int_{-1}^1 (1-t)p(t)w(t)dt}{\int_{-1}^1 p(t)w(t)dt} \geq \\ &\geq \frac{m}{M} \min_{-1 < z_1 < \dots < z_{i-1} < 1} \max_{\substack{p \in S \\ p(z_1) = \dots = p(z_{i-1}) = 0}} \frac{\int_{-1}^1 (1-t)p(t)dt}{\int_{-1}^1 p(t)dt} = \frac{m}{M} (1 - \xi_i^u(0)). \end{aligned}$$

The proof of the other inequalities in (29) is similar. Inequality (30) now follows from (28) and (29).  $\square$

**Proposition 3.6.** *Suppose the weight function  $w$  satisfies  $0 < m \leq w \leq M$  almost everywhere. Let  $x := \xi_i(a)$  for some  $-\infty \leq a \leq \infty$  and  $1 \leq i \leq n$ . Then there exist absolute constants  $C, c > 0$  such that*

$$\sqrt{1+x} \leq C \sqrt{\frac{M}{m}} \cdot \frac{i}{n} \quad \text{and} \quad \sqrt{1-x} \leq C \sqrt{\frac{M}{m}} \cdot \frac{n+1-i}{n}.$$

In addition, if  $x \leq \xi_n(0)$  then

$$\sqrt{1-x} \geq c \sqrt{\frac{m}{M}} \cdot \frac{n+1-i}{n}. \quad (31)$$

Similarly, if  $x \geq \xi_1(0)$  then

$$\sqrt{1+x} \geq c \sqrt{\frac{m}{M}} \cdot \frac{i}{n}.$$

We remark that the condition  $x \leq \xi_n(0)$  is violated if and only if  $i = n$  and  $a > 0$ . Indeed, one cannot expect the estimate (31) to hold uniformly when  $x > \xi_n(0)$  since if  $i = n$  then  $x \rightarrow 1$  as  $a \rightarrow \infty$ . A similar remark holds for the condition  $x \geq \xi_1(0)$ .

*Proof.* The proposition follows from (30) by using (8) to note that if  $a \geq 0$  then  $\xi_i(0) \leq \xi_i(a) \leq \xi_{i+1}(0)$  and if  $a \leq 0$  then  $\xi_{i-1}(0) \leq \xi_i(a) \leq \xi_i(0)$ .  $\square$

### 3.2.2 Weights and distances between nodes

In this section we give upper bounds on the weights and the inter-node distance in the quadrature formulas  $\Sigma_x$ . Recall that  $S_x$  is the set of nodes of  $\Sigma_x$ .

**Lemma 3.7.** *Suppose the weight function  $w$  satisfies  $0 < m \leq w \leq M$  almost everywhere. Then there exists an absolute constant  $C > 0$  such that for every  $-1 < x < 1$ ,*

$$\lambda_x(-1) \leq C \frac{M^2}{m} \cdot \frac{1}{n^2} \quad \text{and} \quad \lambda_x(1) \leq C \frac{M^2}{m} \cdot \frac{1}{n^2}.$$

*Proof.* It suffices to prove the inequality for  $\lambda_x(1)$  as the inequality for  $\lambda_x(-1)$  follows from it by considering the reversed weight function  $\tilde{w}(t) := w(-t)$ . Let  $1 \leq r \leq n$  and  $-\infty \leq a < \infty$  be the

unique numbers for which  $x = \xi_r(a)$ . If  $0 \leq a < \infty$  then  $1 \notin S_x$  and hence  $\lambda_x(1) = 0$  and there is nothing to prove. Suppose  $-\infty \leq a < 0$ . Define the following polynomial

$$q(t) := \left[ \prod_{i=1}^{n-1} \left( \frac{t - \xi_i(a)}{1 - \xi_i(a)} \right)^2 \right] \frac{t - \xi_n(a)}{1 - \xi_n(a)}.$$

Observe that  $\deg q = 2n - 1$  and  $q$  vanishes on  $\xi_i(a)$  for all  $i$ . Now, on the one hand, we may use the quadrature formula  $\Sigma_x$  to obtain that

$$\int_{-1}^1 q(t)w(t)dt = \sum_{u \in S_x} \lambda_x(u)q(u) = \lambda_x(1). \quad (32)$$

On the other hand, since  $q \leq 0$  on  $[-1, \xi_n(a)]$  and  $q \leq 1$  on  $(\xi_n(a), 1]$  it follows that

$$\int_{-1}^1 q(t)w(t)dt \leq (1 - \xi_n(a)) M. \quad (33)$$

The lemma follows by putting (32) and (33) together with the fact that  $1 - \xi_n(a) \leq C \frac{M}{m} \cdot \frac{1}{n^2}$  by Proposition 3.6.  $\square$

**Lemma 3.8.** *Suppose the weight function  $w$  satisfies  $w \leq M$  almost everywhere. Then there exists an absolute constant  $C > 0$  such that*

$$\lambda(x) \leq \frac{CM}{n} \max \left\{ \sqrt{1 - x^2}, \frac{1}{n} \right\}, \quad -1 < x < 1. \quad (34)$$

We remark that this bound is not sharp solely under the condition that  $w \leq M$  almost everywhere. For instance, for the Jacobi weight  $w(x) = (1 - x)^\alpha (1 + x)^\beta$  with  $0 < \alpha, \beta \leq 1/2$  we have that  $\lambda(x)$  is of order  $n^{-(2+2\max\{\alpha, \beta\})}$  near one of the endpoints of the interval [10, (15.3.1), (4.21.7) and Theorem 8.21.13]. However, the bound is sharp up to the value of the constant if one imposes some additional assumptions on  $w$  (in particular, in the cases of interest in our main theorems), see Corollary 5.4.

*Proof.* As in the previous section, we denote by  $u$  the constant weight function,  $u \equiv 1$  on  $[-1, 1]$ . Fix  $-1 < x < 1$ . Let  $S$  be the set of all polynomials  $f$  of degree  $\leq 2n - 1$  that are non-negative on  $[-1, 1]$  and satisfy  $f(x) = 1$ . It is well known that  $\lambda^w(x) = \min_{f \in S} \int_{-1}^1 f(t)w(t)dt$  (see [7, Chapter II, section 4]). Therefore

$$\lambda^w(x) = \min_{f \in S} \int_{-1}^1 f(t)w(t)dt \leq M \min_{f \in S} \int_{-1}^1 f(t)dt = M\lambda^u(x).$$

Thus it suffices to prove (34) for the weight function  $u$ . To this end, let  $1 \leq i \leq n$  and  $-\infty \leq a < \infty$  be the unique numbers for which  $x = \xi_i^u(a)$ . We may assume without loss of generality that  $n \geq 3$  since otherwise the lemma is trivial. We consider separately three cases, in all of which we rely on the Chebyshev-Markov-Stieltjes inequalities (2) and Proposition 3.6.

1. If  $i = 1, 2$ ,

$$\lambda^u(x) \leq \underline{\pi}^u(\xi_3^u(a)) \leq \int_{-1}^{\xi_3^u(a)} dt = \xi_3^u(a) + 1 \leq \frac{C}{n^2}.$$

2. If  $i = n - 1, n$ ,

$$\lambda^u(x) \leq 2 - \pi^u(\xi_{n-2}^u(a)) \leq 2 - \int_{-1}^{\xi_{n-2}^u(a)} dt = 1 - \xi_{n-2}^u(a) \leq \frac{C}{n^2}.$$

3. If  $2 < i < n - 1$ , by (28) we have

$$\begin{aligned} \lambda^u(x) &= \underline{\pi}^u(\xi_{i+1}^u(a)) - \pi^u(\xi_{i-1}^u(a)) \leq \int_{-1}^{\xi_{i+1}^u(a)} dt - \int_{-1}^{\xi_{i-1}^u(a)} dt = \xi_{i+1}^u(a) - \xi_{i-1}^u(a) \leq \\ &\leq \xi_{i+2}^u(0) - \xi_{i-2}^u(0) \leq \cos\left(\frac{2(i-2)-1}{2n+1}\pi\right) - \cos\left(\frac{2(i+2)}{2n+1}\pi\right) \leq \\ &\leq C \frac{i(n+1-i)}{n^3} \leq \frac{C}{n} \sqrt{1-x^2}. \end{aligned} \quad \square$$

The next lemma deduces upper bounds on the inter-node distance in the quadrature formulas  $\Sigma_x$ . In the case  $a = 0$  (from which the general case follows using (8)), these bounds appear in the work of Erdős and Turán [5].

**Lemma 3.9.** *Suppose the weight function  $w$  satisfies  $0 < m \leq w \leq M$  almost everywhere. Then there exists an absolute constant  $C > 0$  such that for every  $1 \leq i \leq n - 1$  and every  $-\infty \leq a \leq \infty$ :*

$$\xi_{i+1}(a) - \xi_i(a) \leq C \left(\frac{M}{m}\right)^2 \frac{i(n-i)}{n^3}. \quad (35)$$

*Proof.* Assume first that  $\xi_i(a), \xi_{i+1}(a) \in (-1, 1)$ . The Chebyshev-Markov-Stieltjes inequalities (2) imply that

$$\begin{aligned} m [\xi_{i+1}(a) - \xi_i(a)] &\leq \int_{\xi_i(a)}^{\xi_{i+1}(a)} w(t) dt = \int_{-1}^{\xi_{i+1}(a)} w(t) dt - \int_{-1}^{\xi_i(a)} w(t) dt \leq \\ &\leq \pi(\xi_{i+1}(a)) - \underline{\pi}(\xi_i(a)) = \lambda(\xi_i(a)) + \lambda(\xi_{i+1}(a)) \end{aligned}$$

and the result follows from Lemma 3.8 and Proposition 3.6. Second, if either  $\xi_i(a)$  or  $\xi_{i+1}(a)$  are in  $\{-1, 1\}$  the result follows from (8).  $\square$

We do not know if the bound (35) is sharp under the conditions of Lemma 3.9, however, the bound is sharp up to a constant depending only on  $w$  if  $w$  satisfies some additional assumptions, e.g., the conditions of Theorem 3.1 with  $\alpha = 0$ . This may be deduced in two different ways from arguments in this paper. First, it follows from Proposition 4.1 below, as follows. If  $a \geq 0$  we may take  $b_+ = 0$  and the limit  $b_- \rightarrow -\infty$  in (37) and use (8). If  $a \leq 0$  we may similarly take  $a_- = 0$  and the limit  $a_+ \rightarrow \infty$  in (36) and use (8). Second, it may be deduced from Lemma 5.2.

## 4 Separation of nodes of quadrature formulas

Part of the motivation for this paper came out of the works [8] and [9] of Kuijlaars. There, a lower bound for  $\pi'$  is established for the special case of a Jacobi weight function  $w(x) = (1-x)^\alpha(1+x)^\beta$  with  $\alpha, \beta \geq 0$  ([8, Proposition 7.2] for the ultraspherical case  $\alpha = \beta$ , [9, Lemma 5.1] for the general case). Another ingredient appearing in those works are results ([8, Proposition 7.3], [9, Lemma 5.2]) bounding the distance between nodes of different canonical representations. In this short section, which is not used in the proofs of our main theorems, we observe that such a result holds also for the more general weight functions which we consider.

**Proposition 4.1.** *Suppose  $w$  satisfies the conditions of Theorem 3.1 with  $\alpha = 0$ . Then there exists a constant  $c(w) > 0$  such that the following holds.*

1. For every  $1 \leq i \leq n$  and  $0 \leq a_- < a_+ < \infty$ ,

$$\xi_i(a_+) - \xi_i(a_-) \geq c(w) \frac{(n+1-i)^2(a_+ - a_-)}{n^3 \left(1 + \frac{n+1-i}{i}a_-\right) \left(1 + \frac{n+1-i}{i}a_+\right)}. \quad (36)$$

2. For every  $1 \leq i \leq n$  and  $-\infty < b_- < b_+ \leq 0$ ,

$$\xi_i(b_+) - \xi_i(b_-) \geq c(w) \frac{i^2(b_+ - b_-)}{n^3 \left(1 + \frac{i}{n+1-i}(-b_-)\right) \left(1 + \frac{i}{n+1-i}(-b_+)\right)}. \quad (37)$$

*Proof.* We only prove (36). The proof of (37) is similar.

Fix  $1 \leq i \leq n$  and  $0 \leq a_- < a_+ < \infty$ . The proof proceeds by bounding  $P_{a_-}(\xi_i(a_+)) - P_{a_-}(\xi_i(a_-))$  from below and from above (alternatively one can bound the same expression with  $P_{a_-}$  replaced by  $P_{a_+}$ ).

On the one hand, by (8) and Lagrange's mean value theorem,

$$P_{a_-}(\xi_i(a_+)) - P_{a_-}(\xi_i(a_-)) = P'_{a_-}(t)(\xi_i(a_+) - \xi_i(a_-))$$

for some  $\xi_i(a_-) < t < \xi_i(a_+)$ . Hence by Proposition 3.3 and Proposition 3.6,

$$\begin{aligned} |P_{a_-}(\xi_i(a_+)) - P_{a_-}(\xi_i(a_-))| &= |P'_{a_-}(t)|(\xi_i(a_+) - \xi_i(a_-)) \leq \\ &\leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^{\frac{3}{2}} \left( 1 + a_- \left( (1-t) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\} + \frac{1}{n} \right) \right) (\xi_i(a_+) - \xi_i(a_-)) \leq \\ &\leq C(w)n \left( \frac{n^2}{i(n+1-i)} \right)^{\frac{3}{2}} \left( 1 + a_- \frac{n+1-i}{i} \right) (\xi_i(a_+) - \xi_i(a_-)). \end{aligned} \quad (38)$$

On the other hand, using the fact that  $P_{a_-}(\xi_i(a_-)) = P_{a_+}(\xi_i(a_+)) = 0$  by definition and substituting the definition (7) of  $P_a$  gives

$$P_{a_-}(\xi_i(a_+)) - P_{a_-}(\xi_i(a_-)) = P_{a_-}(\xi_i(a_+)) - P_{a_+}(\xi_i(a_+)) = (a_+ - a_-)(1 - \xi_i(a_+))\psi(\xi_i(a_+)). \quad (39)$$



Similarly, we may obtain an expression in terms of  $\varphi$  by writing,

$$P_{a_-}(\xi_i(a_+)) - P_{a_-}(\xi_i(a_-)) = P_{a_-}(\xi_i(a_+)) - \frac{a_-}{a_+} P_{a_+}(\xi_i(a_+)) = \frac{a_+ - a_-}{a_+} \varphi(\xi_i(a_+)). \quad (40)$$

Combining (39) and (40), applying Corollary 3.2 and Proposition 3.6 yields

$$\begin{aligned} & \left( a_+ + (1 + \xi_i(a_+)) \min \left\{ n, \frac{1}{\sqrt{1 - \xi_i(a_+)^2}} \right\} \right) |P_{a_-}(\xi_i(a_+)) - P_{a_-}(\xi_i(a_-))| = \\ & = (a_+ - a_-) \left[ |\varphi(\xi_i(a_+))| + (1 - \xi_i(a_+)^2) \min \left\{ n, \frac{1}{\sqrt{1 - \xi_i(a_+)^2}} \right\} |\psi(\xi_i(a_+))| \right] \geq \\ & \geq c(w)(a_+ - a_-) \min \left\{ n, \frac{1}{\sqrt{1 - \xi_i(a_+)^2}} \right\}^{1/2} \geq c(w)(a_+ - a_-) \frac{n}{\sqrt{i(n+1-i)}}. \end{aligned}$$

In addition, by Proposition 3.6,

$$(1 + \xi_i(a_+)) \min \left\{ n, \frac{1}{\sqrt{1 - \xi_i(a_+)^2}} \right\} \leq C(w) \frac{i}{n+1-i}.$$

Putting together the last two inequalities we finally arrive at

$$|P_{a_-}(\xi_i(a_+)) - P_{a_-}(\xi_i(a_-))| \geq c(w) \frac{(a_+ - a_-) \frac{n}{\sqrt{i(n+1-i)}}}{a_+ + \frac{i}{n+1-i}}. \quad (41)$$

Comparing (38) and (41) shows that

$$C(w) \cdot n \cdot \left( \frac{n^2}{i(n+1-i)} \right)^{\frac{3}{2}} \left( 1 + a_- \frac{n+1-i}{i} \right) (\xi_i(a_+) - \xi_i(a_-)) \geq c(w) \frac{(a_+ - a_-) \frac{n}{\sqrt{i(n+1-i)}}}{a_+ + \frac{i}{n+1-i}},$$

from which (36) follows.  $\square$

## 5 Estimating $P'_a$ and the interpolation polynomials

In this section we prove a lower bound on  $|P'_a|$  at roots of  $P_a$ . This lower bound will be used in the next section to prove Theorem 1.1 and Theorem 1.2. We also give bounds for certain interpolation polynomials defined below.

We recall the definition of the sign function from (18). Define also, for real  $x$ , the truncation operation,

$$\bar{x} := \begin{cases} \xi_1(0) & x \leq \xi_1(0) \\ x & x \in [\xi_1(0), \xi_n(0)] \\ \xi_n(0) & x \geq \xi_n(0) \end{cases}. \quad (42)$$

**Lemma 5.1.** *Suppose  $w$  satisfies the conditions of Theorem 3.1 with  $\alpha = 0$ . Then there exists a constant  $c(w) > 0$  such that the following holds for every  $-1 < x < 1$ .*

1. *If  $x = \xi_r(a)$  for  $1 \leq r \leq n$  and  $-\infty < a < \infty$  then*

$$|P'_a(x)| \geq c(w) \sqrt{\frac{n}{\lambda(x)}} \max \left\{ \frac{|a|}{1 + \operatorname{sgn}(a)x}, \frac{1}{\sqrt{1-x^2}} \right\}.$$

2. *If  $x = \eta_r$  for  $1 \leq r \leq n-1$  then*

$$|\psi'(x)| \geq c(w) \sqrt{\frac{n}{\lambda(x)}} \cdot \frac{1}{1-x^2}.$$

Our proof of the lemma relies on lower bounds for  $|\varphi|$  and  $|\psi|$ . The bound (16) shows that  $|\varphi|$  and  $|\psi|$  cannot be simultaneously small. Thus, near a root of one, the other must be large. The next lemma makes this idea precise.

**Lemma 5.2.** *Suppose  $w$  satisfies the conditions of Theorem 3.1 with  $\alpha = 0$ . Then there exists a constant  $c(w) > 0$  such that the following holds.*

1. *For every  $1 \leq r \leq n-1$  the interval  $I := \{x : |x - \eta_r| \leq c(w) \frac{r(n+1-r)}{n^3}\}$  satisfies  $I \subseteq [-1, 1]$  and*

$$\min_{x \in I} |\varphi(x)| \geq c(w) \frac{n}{\sqrt{r(n+1-r)}}.$$

2. *For every  $1 \leq r \leq n$  the interval  $I := \{x : |x - \xi_r(0)| \leq c(w) \frac{r(n+1-r)}{n^3}\}$  satisfies  $I \subseteq [-1, 1]$  and*

$$\min_{x \in I} |\psi(x)| \geq c(w) \left( \frac{n}{\sqrt{r(n+1-r)}} \right)^3.$$

*Proof.* We prove only the first part. The proof of the second part is similar.

Denote  $\rho := \frac{r(n+1-r)}{n^2}$ . Let  $\varepsilon = \varepsilon(w) > 0$  be a constant, depending on  $w$  but independent of  $n$ , whose value is sufficiently small for the following calculations. Define

$$I_1 := \left\{ x : |x - \eta_r| \leq \varepsilon \frac{\rho}{n} \right\} \quad \text{and} \quad I_2 := \left\{ x : 1 - x^2 \geq \frac{1}{2}(1 - \eta_r^2) \right\}.$$

Observe that  $c(w)\rho^2 \leq 1 - \eta_r^2 \leq C(w)\rho^2$  by (8) and Proposition 3.6. Thus, since  $\psi(\eta_r) = 0$ , Corollary 3.2 implies that

$$|\varphi(\eta_r)| \geq c(w) \min \left\{ n, \frac{1}{\sqrt{1 - \eta_r^2}} \right\}^{1/2} \geq \frac{c(w)}{\sqrt{\rho}}.$$

In addition, Proposition 3.3 implies that

$$|\varphi'(x)| \leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{\frac{3}{2}} \leq C(w) \frac{n}{\rho^{3/2}}, \quad x \in I_2.$$

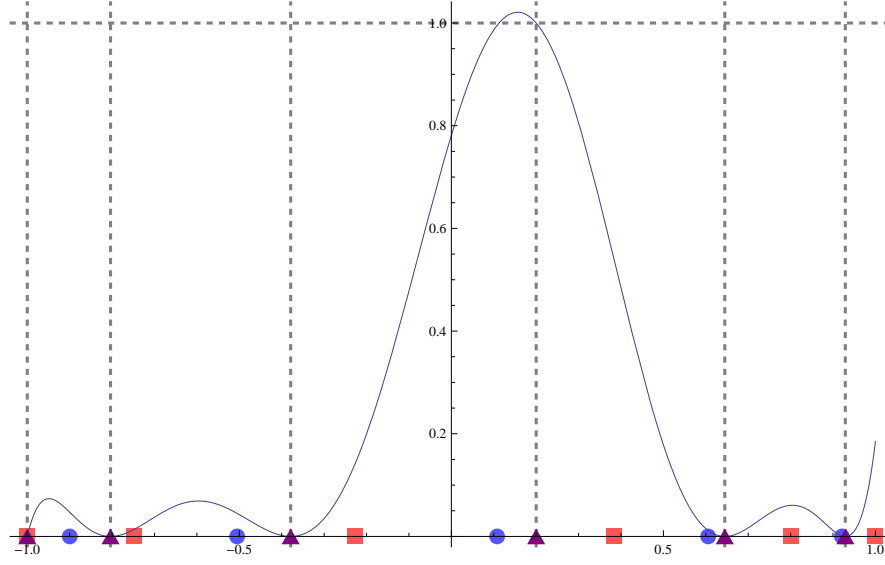


Figure 5: A plot of the polynomial  $q_x$  for  $n = 5$ ,  $x = 0.2$  and the weight function  $w(t) = \max\{1, 1+4t\}$ . The circles on the axis denote the nodes of the Gaussian quadrature, the squares denote the nodes of the Lobatto quadrature and the triangles denote the nodes of the canonical representation  $\Sigma_{0.2}$ . Observe that, by definition,  $q_x(x) = 1$  and  $q'_x$  is zero at all nodes of  $\Sigma_{0.2} - \{-1, x, 1\}$ .

Finally, since  $\rho \geq \frac{1}{n}$ , it follows that  $I_1 \subseteq I_2$  if  $\varepsilon$  is sufficiently small. Thus, for sufficiently small  $\varepsilon$ ,

$$|\varphi(x)| \geq |\varphi(\eta_r)| - |x - \eta_r| \max_{y \in I_2} |\varphi'(y)| \geq \frac{c(w)}{\sqrt{\rho}} - \varepsilon \frac{\rho}{n} C(w) \frac{n}{\rho^{3/2}} \geq \frac{c(w)}{\sqrt{\rho}}, \quad x \in I_1. \quad \square$$

A second tool in our proof of Lemma 5.1 is the following polynomial, which is a relative of the Lagrange interpolation polynomial for  $P_a$ .

Recall that  $S_x$  is the set of nodes of  $\Sigma_x$  and recall the definition of the index function  $I$  from (10). For  $-1 < x < 1$  define the polynomial

$$q_x(t) := \prod_{u \in S_x - \{x\}} \left( \frac{t - u}{x - u} \right)^{I(u)}. \quad (43)$$

Figure 5 shows a plot of  $q_x$  for a certain choice of the parameters. We list some straightforward properties of  $q_x$ .

1.  $\deg(q_x) = \sum_{u \in S_x} I(u) - 2 \leq 2n - 1$ .
2.  $q_x(x) = 1$ ,  $q_x(u) = 0$  for  $u \in S_x - \{x\}$  and  $q'_x(u) = 0$  for  $u \in (S_x - \{x\}) \cap (-1, 1)$ . Furthermore,  $q_x$  is the (unique) least degree polynomial satisfying these equalities.
3.  $q_x \geq 0$  on  $[-1, 1]$ .

4. Writing  $x = \xi_r(a)$  we have the following formula

$$q_x(t) = \begin{cases} \frac{1+\operatorname{sgn}(a)t}{1+\operatorname{sgn}(a)x} \left( \frac{P_a(t)}{(t-x)P'_a(x)} \right)^2 & -\infty < a < \infty \\ \frac{1-t^2}{1-x^2} \left( \frac{\psi(t)}{(t-x)\psi'(x)} \right)^2 & |a| = \infty \end{cases}. \quad (44)$$

Observe that by the first two properties above, using the quadrature formula  $\Sigma_x$ , we have

$$\int_{-1}^1 q_x(t)w(t)dt = \sum_{u \in S_x} \lambda_x(u)q_x(u) = \lambda(x). \quad (45)$$

**Lemma 5.3.** *Suppose the weight function  $w$  satisfies  $0 < m \leq w \leq M$  almost everywhere. Let  $1 \leq r \leq n$  and let  $2 \leq r' \leq n-1$  satisfy  $|r' - r| \leq 1$ . Suppose that  $I$  is an interval satisfying  $I \subseteq [\xi_{r'-1}(0), \xi_{r'+1}(0)]$ . Then,*

1. *For every  $-\infty < a < \infty$ ,*

$$|P'_a(\xi_r(a))| \geq c(w)\sqrt{|I|} \frac{n^3}{r(n+1-r)} \frac{1}{\sqrt{\lambda(\xi_r(a))}} \min_{t \in I} |P_a(t)|. \quad (46)$$

2. *If  $1 \leq r \leq n-1$ ,*

$$|\psi'(\eta_r)| \geq c(w)\sqrt{|I|} \frac{n^3}{r(n+1-r)} \frac{1}{\sqrt{\lambda(\eta_r)}} \min_{t \in I} |\psi(t)|.$$

The inequalities in the lemma may be trivial, in the sense that their right-hand sides may vanish, but we will avoid this possibility in our usage by choosing  $I$  appropriately.

*Proof.* We prove only the first part of the lemma. The second part is similar. Observe that by (45), the non-negativity of  $q_{\xi_r(a)}$  and our assumption that  $w$  is bounded below, we have

$$\lambda(\xi_r(a)) \geq \int_I q_{\xi_r(a)}(t)w(t)dt \geq |I| \min_{t \in I} (q_{\xi_r(a)}(t)w(t)) \geq c(w)|I| \min_{t \in I} q_{\xi_r(a)}(t). \quad (47)$$

We continue to estimate the minimum. By (44),

$$q_{\xi_r(a)}(t) = \frac{1 + \operatorname{sgn}(a)t}{1 + \operatorname{sgn}(a)\xi_r(a)} \cdot \frac{1}{(t - \xi_r(a))^2} \cdot P_a(t)^2 \cdot \frac{1}{P'_a(\xi_r(a))^2}. \quad (48)$$

Let us estimate each of the first two factors above separately. First, observe that by the conditions of the lemma and Proposition 3.6,

$$\min_{t \in I} \frac{1 + \operatorname{sgn}(a)t}{1 + \operatorname{sgn}(a)\xi_r(a)} \geq c(w). \quad (49)$$

Second, note that by the conditions of the lemma and Lemma 3.9,

$$\max_{t \in I} |t - \xi_r(a)| \leq \max\{|\xi_{r'+1}(0) - \xi_r(a)|, |\xi_{r'-1}(0) - \xi_r(a)|\} \leq C(w) \frac{r(n+1-r)}{n^3}. \quad (50)$$

Substituting (49) and (50) into (48) we obtain

$$\min_{t \in I} q_{\xi_r(a)}(t) \geq c(w) \left( \frac{n^3}{r(n+1-r)} \right)^2 \frac{1}{P'_a(\xi_r(a))^2} \left( \min_{t \in I} |P_a(t)| \right)^2.$$

Finally, we may continue (47) to obtain

$$\lambda(\xi_r(a)) \geq c(w)|I| \left( \frac{n^3}{r(n+1-r)} \right)^2 \cdot \frac{1}{P'_a(\xi_r(a))^2} \left( \min_{t \in I} |P_a(t)| \right)^2$$

and the lemma follows.  $\square$

*Proof of Lemma 5.1.* We prove only the first part of the lemma. The second part is similar and even simpler. To simplify the presentation, take  $2 \leq r' \leq n-1$  such that  $|r' - r| \leq 1$  (when  $n = 1, 2$  the definitions of  $\tilde{I}_\varphi$  and  $\tilde{I}_\psi$  below may be adjusted properly and the proof of Lemma 5.3 repeated for them. We omit the details). By Lemma 5.2 there is a constant  $c(w) > 0$  such that

$$\min_{x \in I_\varphi} |\varphi(x)| \geq c(w) \frac{n}{\sqrt{r'(n+1-r')}}, \quad (51)$$

$$\min_{x \in I_\psi} |\psi(x)| \geq c(w) \left( \frac{n}{\sqrt{r'(n+1-r')}} \right)^3. \quad (52)$$

where

$$I_\varphi := \left\{ x : |x - \eta_{r'}| \leq c(w) \frac{r'(n+1-r')}{n^3} \right\},$$

$$I_\psi := \left\{ x : |x - \xi_{r'}(0)| \leq c(w) \frac{r'(n+1-r')}{n^3} \right\}.$$

In particular,  $\varphi$  does not change sign in  $I_\varphi$  and  $\psi$  does not change sign in  $I_\psi$ . Consequently, since  $\eta_{r'} \in I_\varphi$  and  $\xi_{r'}(0) \in I_\psi$ , then necessarily

$$I_\varphi \subseteq [\xi_{r'}(0), \xi_{r'+1}(0)] \quad \text{and} \quad I_\psi \subseteq [\eta_{r'-1}, \eta_{r'}]. \quad (53)$$

Thus  $\psi$  changes sign in  $I_\varphi$  exactly once, at the midpoint  $\eta_{r'}$ , and  $\varphi$  changes sign in  $I_\psi$  exactly once, at the midpoint  $\xi_{r'}(0)$ .

Let  $\tilde{I}_\varphi$  be the sub-segment of  $I_\varphi$  in which  $\varphi$  and  $a\psi$  have opposite signs. Precisely,

$$\tilde{I}_\varphi := \{x \in I_\varphi : a\varphi(x)\psi(x) \leq 0\}.$$

(the sub-segment to the left of  $\eta_{r'}$  if  $a < 0$  or the sub-segment to the right of  $\eta_{r'}$  if  $a > 0$  or the whole  $I_\varphi$  if  $a = 0$ , see Figure 4). In the same manner, let  $\tilde{I}_\psi$  be the sub-segment of  $I_\psi$  in which  $\varphi$  and  $a\psi$  have opposite signs. Then, using (7) and (51) and the fact that  $|r' - r| \leq 1$ ,

$$\min_{t \in \tilde{I}_\varphi} |P_a(t)| \geq \min_{t \in \tilde{I}_\varphi} |\varphi(t)| \geq c(w) \frac{n}{\sqrt{r(n+1-r)}} \quad (54)$$

and, similarly, using (7), (52) and (53),

$$\begin{aligned} \min_{t \in \tilde{I}_\psi} |P_a(t)| &\geq |a| \min_{t \in \tilde{I}_\psi} (1 - \operatorname{sgn}(a)t) |\psi(t)| \geq \\ &\geq c(w) |a| (1 - \max\{\operatorname{sgn}(a)\eta_{r'-1}, \operatorname{sgn}(a)\eta_{r'}\}) \left( \frac{n}{\sqrt{r(n+1-r)}} \right)^3. \end{aligned} \quad (55)$$

Observe that by our definitions,

$$\min(|\tilde{I}_\varphi|, |\tilde{I}_\psi|) \geq c(w) \frac{r(n+1-r)}{n^3}.$$

Note also that  $\tilde{I}_\varphi$  and  $\tilde{I}_\psi$  satisfy the assumptions of Lemma 5.3 by (53). Thus, by plugging (54) and (55), respectively, in (46), and using Proposition 3.6, we have that

$$\begin{aligned} |P'_a(\xi_r(a))| &\geq c(w) \frac{n}{\sqrt{r(n+1-r)}} \sqrt{\frac{n}{\lambda(\xi_r(a))}} \min_{t \in \tilde{I}_\varphi} |P_a(t)| \geq \\ &\geq c(w) \frac{n^2}{r(n+1-r)} \cdot \sqrt{\frac{n}{\lambda(\xi_r(a))}} \geq c(w) \sqrt{\frac{n}{\lambda(\xi_r(a))}} \cdot \frac{1}{\sqrt{1 - \xi_r(a)^2}} \end{aligned} \quad (56)$$

and

$$\begin{aligned} |P'_a(\xi_r(a))| &\geq c(w) \frac{n}{\sqrt{r(n+1-r)}} \sqrt{\frac{n}{\lambda(\xi_r(a))}} \min_{t \in \tilde{I}_\psi} |P_a(t)| \geq \\ &\geq c(w) \left( \frac{n^2}{r(n+1-r)} \right)^2 \sqrt{\frac{n}{\lambda(\xi_r(a))}} |a| (1 - \max\{\operatorname{sgn}(a)\eta_{r'-1}, \operatorname{sgn}(a)\eta_{r'}\}) \geq \\ &\geq c(w) \sqrt{\frac{n}{\lambda(\xi_r(a))}} \cdot \frac{|a|}{1 + \operatorname{sgn}(a)\xi_r(a)}. \end{aligned} \quad (57)$$

The lemma follows by putting together (57) and (56).  $\square$

It is useful to note that combining Lemma 5.1 with Proposition 3.3 we may obtain a lower bound on the function  $\lambda(x)$ , matching the bound given by Lemma 3.8 up to a constant depending on  $w$ . This is embodied in the following corollary which is probably well-known to experts in the field.

**Corollary 5.4.** *Suppose  $w$  satisfies the conditions of Theorem 3.1 with  $\alpha = 0$ . Then there exists a constant  $c(w) > 0$  such that*

$$\lambda(x) \geq \frac{c(w)}{n} \max \left\{ \sqrt{1 - x^2}, \frac{1}{n} \right\}, \quad -1 < x < 1. \quad (58)$$

*Proof.* Suppose  $x = \xi_r(a)$  for some  $1 \leq r \leq n$  and  $-\infty < a < \infty$ . By Lemma 5.1,

$$|P'_a(x)| \geq c(w) \sqrt{\frac{n}{\lambda(x)}} \max \left\{ \frac{|a|}{1 + \operatorname{sgn}(a)x}, \frac{1}{\sqrt{1 - x^2}} \right\}. \quad (59)$$

Observe that by Proposition 3.6,  $\min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\} \leq \frac{C(w)}{\sqrt{1-x^2}}$ . Thus, by Proposition 3.3,

$$\begin{aligned} |P'_a(x)| &\leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{3/2} \left( 1 + |a| \left( (1 - \operatorname{sgn}(a)x) \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\} + \frac{1}{n} \right) \right) \leq \\ &\leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{1/2} \left( \frac{1}{\sqrt{1-x^2}} + |a| \left( \frac{1 - \operatorname{sgn}(a)x}{1-x^2} + 1 \right) \right) \leq \\ &\leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{1/2} \max \left\{ \frac{1}{\sqrt{1-x^2}}, \frac{|a|}{1 + \operatorname{sgn}(a)x} \right\}. \end{aligned}$$

Combined with (59) this implies (58). Now suppose  $x = \eta_r$  for some  $1 \leq r \leq n-1$ . Then the result follows since by Lemma 5.1,

$$|\psi'(x)| \geq c(w) \sqrt{\frac{n}{\lambda(x)}} \cdot \frac{1}{1-x^2},$$

and by Proposition 3.3,

$$|\psi'(x)| \leq C(w)n \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{5/2} \leq C(w) \frac{n}{1-x^2} \cdot \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{1/2}. \quad \square$$

## 5.1 Estimating the interpolation polynomials

In this section we explore the localization properties of the interpolation polynomials  $q_x$ . Specifically, we show in the next proposition that  $q_x$  is everywhere bounded by a constant and is in fact much smaller away from the point  $x$ . The proof of Theorem 1.1 does not require the results of this section but these results are used in the proofs of Theorem 1.2 and Theorem 1.3.

**Proposition 5.5.** *Suppose  $w$  satisfies the conditions of Theorem 3.1 with  $\alpha = 0$ . Then there exists a constant  $C(w) > 0$  such that for any  $-1 < t, x < 1$ ,  $x \neq t$ , the following bounds hold*

$$q_x(t) \leq C(w), \quad (60)$$

and

$$q_x(t) \leq \frac{C(w)}{n \max \left\{ 1, n\sqrt{1-t^2} \right\} (t-x)^2}. \quad (61)$$

We start with the following lemma which is inspired by Erdős and Lengyel [4].

**Lemma 5.6.** *Suppose the weight function  $w$  satisfies  $0 < m \leq w \leq M$  almost everywhere. Then for every  $-1 < t, x < 1$ ,*

$$q_x(t) \leq \frac{M}{m} \cdot \frac{1 + \operatorname{sgn}(x-t)x}{1 + \operatorname{sgn}(x-t)t}.$$

*Proof.* We show the proof for  $t \leq x$ . The proof for  $x \leq t$  is similar. Let  $g(u) := q_x \left( -1 + \frac{1+t}{1+x}(1+u) \right)$ . Since  $\deg g = \deg q_x \leq 2n-1$  and  $g, q_x \geq 0$  on  $[-1, 1]$  we get, using the quadrature formula  $\Sigma_x$ , and (45),

$$\begin{aligned} \lambda(x)q_x(t) &= \lambda(x)g(x) \leq \sum_{u \in S_x} \lambda_x(u)g(u) = \int_{-1}^1 g(s)w(s)ds = \\ &= \int_{-1}^1 q_x \left( -1 + \frac{1+t}{1+x}(1+s) \right) w(s)ds = \frac{1+x}{1+t} \int_{-1}^{1-2\frac{x-t}{1+x}} q_x(\sigma)w \left( -1 + \frac{1+x}{1+t}(1+\sigma) \right) d\sigma \leq \\ &\leq \frac{1+x}{1+t} \cdot \frac{M}{m} \int_{-1}^{1-2\frac{x-t}{1+x}} q_x(\sigma)w(\sigma)d\sigma \leq \frac{1+x}{1+t} \cdot \frac{M}{m} \int_{-1}^1 q_x(\sigma)w(\sigma)d\sigma = \frac{1+x}{1+t} \cdot \frac{M}{m} \lambda(x). \quad \square \end{aligned}$$

*Proof of Proposition 5.5.* We divide into three cases.

1. Suppose  $x = \xi_r(a)$  for some  $1 \leq r \leq n$  and  $|a| \leq \frac{1}{(1-\operatorname{sgn}(a)t) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}}$ .

Let  $t_0 = \eta_{r'}$  where  $1 \leq r' \leq n-1$  is such that  $|r' - r| \leq 1$ . By Theorem 3.1 and Proposition 3.6,

$$|P_a(t_0)| = |\varphi(t_0)| = |\varphi(t_0)| + \sqrt{1-t_0^2}|\psi(t_0)| \geq \frac{c(w)}{(1-t_0^2)^{1/4}}$$

and by Corollary 3.2,

$$\begin{aligned} |P_a(t)| &\leq C(w) \max \left\{ \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^{1/2}, |a|(1-\operatorname{sgn}(a)t) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^{3/2} \right\} = \\ &= C(w) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^{1/2}. \end{aligned}$$

Therefore, by (44),

$$\frac{q_x(t)}{q_x(t_0)} = \frac{1+\operatorname{sgn}(a)t}{1+\operatorname{sgn}(a)t_0} \left( \frac{P_a(t)(t_0-x)}{P_a(t_0)(t-x)} \right)^2 \leq C(w) \frac{(1+\operatorname{sgn}(a)t) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\} (t_0-x)^2}{(1+\operatorname{sgn}(a)t_0) \frac{1}{\sqrt{1-t_0^2}} (t-x)^2}.$$

Hence, using Proposition 3.6, Lemma 5.6 and Lemma 3.9 we get

$$\begin{aligned} q_x(t) &= \frac{q_x(t)}{q_x(t_0)} q_x(t_0) \leq C(w) \frac{(1+\operatorname{sgn}(a)t) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\} (t_0-x)^2}{(1+\operatorname{sgn}(a)t_0) \frac{1}{\sqrt{1-t_0^2}} (t-x)^2} \leq \\ &\leq C(w) \frac{(1+\operatorname{sgn}(a)t) \max \left\{ \frac{1}{n^2}, 1-\operatorname{sgn}(a)x \right\} \max \left\{ \frac{1}{n}, \sqrt{1-x^2} \right\}}{n \max \left\{ 1, n\sqrt{1-t^2} \right\} (t-x)^2}. \end{aligned} \quad (62)$$

The bound (61) follows immediately.



To obtain (60) assume first that  $-1 + \frac{1}{4n^2} \leq x \leq 1 - \frac{1}{4n^2}$ . By (62),

$$q_x(t) \leq \frac{C(w)}{\max \left\{ 1, n\sqrt{1-t^2} \right\}} \frac{(1 + \operatorname{sgn}(a)t)(1 - \operatorname{sgn}(a)x)}{|t-x|} \cdot \frac{1-x^2}{|t-x|}.$$

We deduce that  $q_x(t) \leq C(w)$  when  $t \notin (\frac{x-1}{2}, \frac{x+1}{2})$  since then  $|t-x| \geq \frac{1}{4}(1 + \operatorname{sgn}(a)t)(1 - \operatorname{sgn}(a)x)$  and  $|t-x| \geq \frac{1-x^2}{4}$ . If  $t \in (\frac{x-1}{2}, \frac{x+1}{2})$  then  $q_x(t) \leq C(w)$  by Lemma 5.6.

Assume now that  $1 - \frac{1}{4n^2} < x < 1$  (the case  $-1 < x < -1 + \frac{1}{4n^2}$  is treated similarly). Let  $M$  be the maximum of  $q_x$  on  $[-1, 1]$  and suppose it is obtained in  $t_1$ . If  $t_1 < \frac{x-1}{2}$ , then by (61),

$$M = q_x(t_1) \leq \frac{C(w)}{n \max \left\{ 1, n\sqrt{1-t_1^2} \right\} (t_1 - x)^2} \leq \frac{C(w)}{n} \leq C(w).$$

If  $\frac{x-1}{2} \leq t_1 \leq \frac{x+1}{2}$  then  $M = q_x(t_1) \leq C(w)$  by Lemma 5.6. Finally, if  $t_1 > \frac{x+1}{2}$  then, since  $|q'_x| \leq (2n-1)^2 M$  everywhere in  $(-1, 1)$  by Markov's inequality (as in (22)),

$$\left| q_x \left( \frac{x+1}{2} \right) \right| \geq |q_x(t_1)| - (2n-1)^2 M \left( t_1 - \frac{x+1}{2} \right) \geq M - (2n-1)^2 M \frac{1-x}{2} \geq \frac{M}{2}$$

and since we have already seen that  $|q_x(\frac{x+1}{2})| \leq C(w)$  we get that  $M \leq 2C(w)$ .

2. Suppose  $x = \xi_r(a)$  for some  $1 \leq r \leq n$  and  $|a| > \frac{1}{(1 - \operatorname{sgn}(a)t) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}}$ .

Let  $t_0 = \xi_r(0)$ . By Theorem 3.1 and Proposition 3.6,

$$|P_a(t_0)| = |a|(1 - \operatorname{sgn}(a)t_0)|\psi(t_0)| \geq \frac{c(w)|a|(1 - \operatorname{sgn}(a)t_0)}{(1 - t_0^2)^{3/4}}$$

and by Corollary 3.2,

$$\begin{aligned} |P_a(t)| &\leq C(w) \max \left\{ \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^{1/2}, |a|(1 - \operatorname{sgn}(a)t) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^{3/2} \right\} = \\ &= C(w) |a|(1 - \operatorname{sgn}(a)t) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^{3/2}. \end{aligned}$$

Therefore, by (44),

$$\frac{q_x(t)}{q_x(t_0)} = \frac{1 + \operatorname{sgn}(a)t}{1 + \operatorname{sgn}(a)t_0} \left( \frac{P_a(t)(t_0 - x)}{P_a(t_0)(t - x)} \right)^2 \leq C(w) \frac{(1 - \operatorname{sgn}(a)t)(1 - t^2) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^3 (t_0 - x)^2}{(1 - \operatorname{sgn}(a)t_0) \frac{1}{\sqrt{1-t_0^2}} (t - x)^2}.$$

Hence, using Proposition 3.6, Lemma 5.6 and Lemma 3.9 we get

$$\begin{aligned} q_x(t) &= \frac{q_x(t)}{q_x(t_0)} q_x(t_0) \leq C(w) \frac{(1 - \operatorname{sgn}(a)t)(1 - t^2) \min \left\{ n, \frac{1}{\sqrt{1-t^2}} \right\}^3 (t_0 - x)^2}{(1 - \operatorname{sgn}(a)t_0) \frac{1}{\sqrt{1-t_0^2}} (t - x)^2} \leq \\ &\leq C(w) \frac{(1 - \operatorname{sgn}(a)t)(1 + \operatorname{sgn}(a)x) \frac{1-t^2}{\max \left\{ \frac{1}{n^2}, 1-t^2 \right\}} \max \left\{ \frac{1}{n}, \sqrt{1-x^2} \right\}}{n \max \left\{ 1, n\sqrt{1-t^2} \right\} (t - x)^2}. \end{aligned} \tag{63}$$

The bound (61) follows immediately.

To obtain (60), first note that if  $t \in (\frac{x-1}{2}, \frac{x+1}{2})$  then  $q_x(t) \leq C(w)$  by Lemma 5.6. Second note that if  $t \notin (\frac{x-1}{2}, \frac{x+1}{2})$  then  $|t - x| \geq \frac{1}{4}(1 - \operatorname{sgn}(a)t)(1 + \operatorname{sgn}(a)x)$ ,  $|t - x| \geq \frac{1-x^2}{4}$  and  $|t - x| \geq \frac{1-t^2}{2}$ . Thus, using (63),

$$q_x(t) \leq C(w) \frac{n\sqrt{1-t^2} \max\left\{\frac{1}{n}, \sqrt{1-x^2}\right\}}{\max\left\{1, n\sqrt{1-t^2}\right\} \sqrt{|t-x|}} \leq C(w) \frac{\max\left\{\sqrt{1-t^2}, \sqrt{1-x^2}\right\}}{\sqrt{|t-x|}} \leq C(w).$$

3. Suppose  $x = \eta_r$  for some  $1 \leq r \leq n-1$ .

This case follows from case 2 above by noting that for every  $-1 < t < 1$ ,

$$q_{\eta_r}(t) = \frac{1-\eta_r}{1-t} \lim_{a \rightarrow \infty} q_{\xi_r(a)}(t) = \frac{1+\eta_r}{1+t} \lim_{a \rightarrow -\infty} q_{\xi_{r+1}(a)}(t). \quad \square$$

## 6 Proof of Theorem 1.1 and Theorem 1.2

In this section we prove Theorem 1.1 and Theorem 1.2. We first explain how the proposition follows from several lemmas. The proof of these lemmas is delayed to the next subsections.

Recall the definition (10) of the index function. In a similar manner to the definition (43) of the polynomial  $q_x$ , we define, for each  $-1 < x < 1$ , the polynomial  $p_x$  to be the unique polynomial satisfying the following properties:

$$\deg(p_x) \leq \sum_{u \in S_x} I(x) - 2, \quad (64)$$

$$p_x(u) = \begin{cases} 1 & u \in S_x \cap [-1, x] \\ 0 & u \in S_x \cap (x, 1] \end{cases}, \quad (65)$$

$$p'_x(u) = 0, \quad u \in (S_x - \{x\}) \cap (-1, 1). \quad (66)$$

We note that  $\deg(p_x) = \sum_{u \in S_x} I(x) - 2$  unless  $x \geq \xi_n(0)$ , in which case  $p_x \equiv 1$ . Figure 6 shows the graph of this polynomial as well as the polynomial  $\underline{p}_x$  defined below. Using the quadrature formula  $\Sigma_x$  and the fact that  $\deg(p_x) \leq 2n-1$  it follows immediately that

$$\int_{-1}^1 p_x(t) w(t) dt = \pi(x).$$

Our first lemma relates the quantity  $\pi'$ , which we would like to estimate, to the polynomial  $p_x$ .

**Lemma 6.1.** *For every weight function  $w$  and every differentiability point  $x \in (-1, 1)$  of  $\pi$  we have*

$$\pi'(x) = -\lambda(x)p'_x(x) = \lambda_x(-1)p'_x(-1) + \lambda_x(1)p'_x(1) - \int_{-1}^1 p'_x(t)w(t)dt. \quad (67)$$

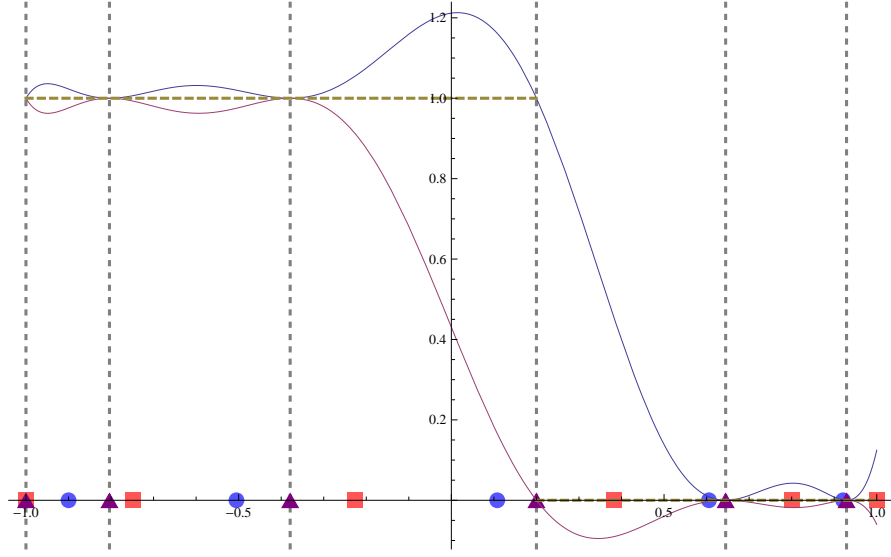


Figure 6: A plot of the polynomials  $p_x$  and  $\underline{p}_x$  for  $n = 5$ ,  $x = 0.2$  and the weight function  $w(t) = \max\{1, 1 + 4t\}$ . Observe that  $p_x$  lies above and  $\underline{p}_x$  lies below the indicator function of the interval  $[0, x]$ , as proved in Lemma 6.5. The circles on the axis denote the nodes of the Gaussian quadrature, the squares denote the nodes of the Lobatto quadrature and the triangles denote the nodes of the quadrature formula  $\Sigma_{0,2}$ .

Our next lemmas estimate the terms on the right-hand side of (67).

**Lemma 6.2.** *Suppose  $w$  satisfies the conditions of Theorem 3.1 with  $\alpha = 0$ . Then there exists a constant  $C(w) > 0$  such that for every  $-1 < x < \xi_n(0)$ ,*

$$\lambda_x(-1)p'_x(-1) + \lambda_x(1)p'_x(1) \geq p_x(1)w(1) - C(w)\frac{\lambda(x)}{1-x}, \quad (68)$$

and for every  $-1 < x < 1$ ,

$$\lambda_x(-1)p'_x(-1) + \lambda_x(1)p'_x(1) \leq -(p_x(-1) - 1)w(-1) + C(w)\lambda(x) \min\left\{\frac{1}{1+x}, n^2\right\}. \quad (69)$$

**Lemma 6.3.** *Suppose  $w$  is an absolutely continuous weight function. For every  $-1 < x < 1$ ,*

$$-(p_x(-1) - 1)w(-1) - \int_{-1}^1 q_x(t)|w'(t)|dt \leq \int_{-1}^1 p'_x(t)w(t)dt + w(x) \leq p_x(1)w(1) + \int_{-1}^1 q_x(t)|w'(t)|dt.$$

**Lemma 6.4.** *1. Suppose  $w$  satisfies the assumptions of Theorem 1.1, and let  $R$  and  $m$  be the constants from these assumptions. Then for every  $-1 < x < 1$ ,*

$$\int_{-1}^1 q_x(t)|w'(t)|dt \leq \frac{R}{m}\lambda(x).$$

2. Suppose  $w$  satisfies the assumptions of Theorem 1.2, and let  $p$  be the constant from these assumptions. Then for every  $-1 < x < 1$ ,

$$\int_{-1}^1 q_x(t) |w'(t)| dt \leq C(w) \lambda(x)^{1-\frac{1}{p}}.$$

We are now prepared to prove Theorem 1.1 and Theorem 1.2.

*Proof of Theorem 1.1 and Theorem 1.2.* We prove only the lower bounds. The proof of the upper bounds is similar and slightly simpler.

When  $x \geq \xi_n(0)$  the bounds follow by taking  $C(w)$  large enough, using the fact that  $\pi$  is non-decreasing by (1) so that  $\pi'(x) \geq 0$ , using Corollary 5.4 and using Proposition 3.6 to see that  $\frac{\lambda(x)}{1-x} \geq c(w)$ . Combining Lemmas 6.1, 6.2 and 6.3,

$$\pi'(x) - w(x) \geq -C(w) \frac{\lambda(x)}{1-x} - \int_{-1}^1 q_x(t) |w'(t)| dt$$

for every differentiability point  $-1 < x < \xi_n(0)$  of  $\pi$ . The stated lower bounds now follow from Lemma 6.4.  $\square$

In the next subsections we prove the above lemmas. To this aim we introduce a second polynomial  $\underline{p}_x$ , whose properties we now explain (see Figure 6).

We define, for each  $-1 < x < 1$ , the polynomial  $\underline{p}_x$  to be the unique polynomial satisfying the following properties:

$$\deg(\underline{p}_x) \leq \sum_{u \in S_x} I(x) - 2, \tag{70}$$

$$\underline{p}_x(u) = \begin{cases} 1 & u \in S_x \cap [-1, x) \\ 0 & u \in S_x \cap [x, 1] \end{cases}, \tag{71}$$

$$\underline{p}'_x(u) = 0, \quad u \in (S_x - \{x\}) \cap (-1, 1). \tag{72}$$

Here, again,  $\deg(\underline{p}_x) = \sum_{u \in S_x} I(x) - 2$  unless  $x \leq \xi_1(0)$ , in which case  $\underline{p}_x \equiv 0$ .

Observe that  $p_x - \underline{p}_x$  is a polynomial of degree  $\leq \sum_{u \in S_x} I(x) - 2$  satisfying  $(p_x - \underline{p}_x)(x) = 1$ ,  $(p_x - \underline{p}_x)(u) = 0$  for every  $u \in S_x - \{x\}$ , and  $(p_x - \underline{p}_x)'(u) = 0$  for every  $u \in (S_x - \{x\}) \cap (-1, 1)$ . Comparing with the properties of  $q_x$  following (43) we conclude that (see also Figures 5 and 6)

$$p_x - \underline{p}_x = q_x. \tag{73}$$

We write  $\chi_A$  for the characteristic function of the set  $A$ .

**Lemma 6.5.** *Let  $w$  be any weight function and let  $-1 < x < 1$ . Then*

$$\underline{p}_x(t) \leq \chi_{[-1, x)}(t) \leq \chi_{[-1, x]}(t) \leq p_x(t) \quad -1 \leq t \leq 1.$$

In addition,

$$\begin{aligned}
\underline{p}'_x(-1) &\leq 0 \leq p'_x(-1) && \text{when } -1 \in S_x, \\
p'_x(-1) &\leq 0 \leq \underline{p}'_x(-1) && \text{when } -1 \notin S_x, \\
p'_x(1) &\leq 0 \leq \underline{p}'_x(1) && \text{when } 1 \in S_x, \\
\underline{p}'_x(1) &\leq 0 \leq p'_x(1) && \text{when } 1 \notin S_x.
\end{aligned}$$

*Proof.* If  $p_x \equiv 1$ , i.e.,  $x \geq \xi_n(0)$ , the claims relating to it are trivial. Otherwise, by Rolle's theorem,  $p'_x$  vanishes at some point (strictly) between any two consecutive points of  $S_x \cap [-1, x]$  and any two consecutive points of  $S_x \cap (x, 1]$ . Together with the points of  $(S_x - \{x\}) \cap (-1, 1)$  we obtain  $\sum_{u \in S_x} I(x) - 3$  distinct points in which  $p'_x$  vanishes. Since  $p'_x$  is a polynomial of degree  $\sum_{u \in S_x} I(x) - 3$  we conclude that these are all the points in which it vanishes, and that it changes sign in each of them and in no other point. The statements concerning  $p_x$  now follow since, by definition,  $p_x(x) = 1 > 0$ .

The statements concerning  $\underline{p}_x$  follow either by using a similar argument, or by noting that  $\underline{p}_x(t) = 1 - \tilde{p}_{-x}(-t)$  where  $\tilde{p}$  is the polynomial  $p$  defined with respect to the reversed weight function  $\tilde{w}(t) := w(-t)$ .  $\square$

## 6.1 Proof of Lemma 6.1

Fix  $-1 < x < 1$  to be a differentiability point of  $\pi$  (recall from Section 2 that  $\pi$  is differentiable at all but finitely many points of  $(-1, 1)$ ). For the proof of the lemma, we generalize the definition of the polynomial  $p_x$  to a one-parameter family of polynomials  $p_x(y, \cdot)$ . Let  $U \subset (-1, 1)$  be an open interval containing  $x$  and not containing any other node of the quadrature formula  $\Sigma_x$ . For every  $y \in U$  we let  $p_x(y, \cdot)$  be the unique polynomial satisfying the following properties:

$$\begin{aligned}
\deg(p_x(y, \cdot)) &\leq \sum_{u \in S_x} I(x) - 2, \\
p_x(y, u) &= \begin{cases} 1 & u \in (S_x - \{x\}) \cap [-1, y) \\ 1 & u = y \\ 0 & (S_x - \{x\}) \cap (y, 1] \end{cases}, \\
p'_x(y, u) &= 0, \quad u \in (S_x - \{x\}) \cap (-1, 1).
\end{aligned} \tag{74}$$

As for  $p_x$ ,  $\deg(p_x(y, \cdot)) = \sum_{u \in S_x} I(x) - 2$  unless  $x \geq \xi_n(0)$ , in which case  $p_x(y, \cdot) \equiv 1$ . The graph of this polynomial is shown in Figure 7. With this definition,  $p_x = p_x(x, \cdot)$ . Clearly,  $\deg p_x(y, \cdot) \leq 2n - 1$ . In addition, it is not difficult to see that  $p_x(y, \cdot) \geq \chi_{[-1, y]}$  in  $[-1, 1]$  in the same manner as in the proof of Lemma 6.5. Thus, applying the quadrature formula  $\Sigma_y$ ,

$$\int_{-1}^1 p_x(y, t) w(t) dt = \sum_{u \in S_y} \lambda_y(u) p_x(y, u) \geq \sum_{u \in S_y \cap [-1, y]} \lambda_y(u) = \pi(y), \quad y \in U \tag{75}$$

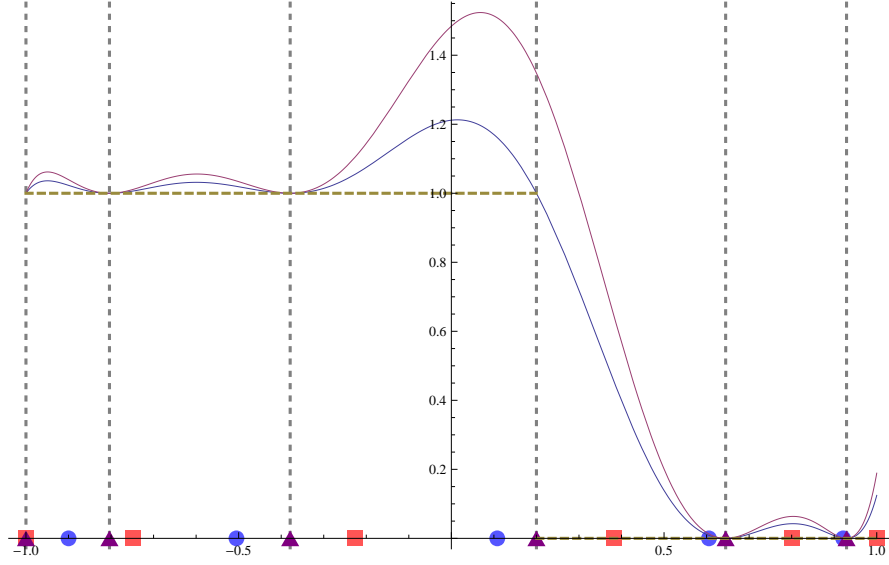


Figure 7: A plot of the polynomials  $p_x(\cdot)$  and  $p_x(y, \cdot)$  for  $n = 5$ ,  $x = 0.2$ ,  $y = 0.3$  and the weight function  $w(t) = \max\{1, 1 + 4t\}$ . The circles on the axis denote the nodes of the Gaussian quadrature, the squares denote the nodes of the Lobatto quadrature and the triangles denote the nodes of the quadrature formula  $\Sigma_{0,2}$ .

with equality when  $y = x$ . We claim that it follows that

$$\pi'(x) = \left( \frac{d}{dy} \int_{-1}^1 p_x(y, t) w(t) dt \right) \Big|_{y=x} = \int_{-1}^1 \frac{\partial p_x}{\partial y}(x, t) w(t) dt. \quad (76)$$

To see this, observe first that  $p_x(y, t)$  is a continuous rational function of  $y$  and  $t$  in the rectangle  $U \times [-1, 1]$ . Thus, the validity of the differentiation under the integral sign follows from the bounded convergence theorem. Second, note that the first equality in (76) follows from the fact that (75) holds in an *open* neighborhood of  $x$ , with equality at  $x$ .

Since  $p_x(y, \cdot)$  is a polynomial of degree at most  $2n - 1$ , it follows that also  $\frac{\partial p_x}{\partial y}(x, \cdot)$  is a polynomial of degree at most  $2n - 1$ , so we can use the quadrature formula  $\Sigma_x$  to calculate

$$\int_{-1}^1 \frac{\partial p_x}{\partial y}(x, t) w(t) dt = \sum_{u \in S_x} \lambda_x(u) \frac{\partial p_x}{\partial y}(x, u). \quad (77)$$

Now observe that for each  $u \in S_x - \{x\}$ , the definition (74) implies that  $p_x(y, u)$  is constant when  $y \in U$ . Thus all terms involving  $u \neq x$  in the right-hand side of (77) vanish. Using (76), we conclude that

$$\pi'(x) = \lambda(x) \frac{\partial p_x}{\partial y}(x, x). \quad (78)$$

Now note that, since  $p_x(z, z)$  is identically 1 by (74), then by the chain rule:

$$0 = \frac{dp_x}{dz}(z, z) = \frac{\partial p_x}{\partial y}(z, z) + \frac{\partial p_x}{\partial u}(z, z).$$

In particular,

$$\frac{\partial p_x}{\partial y}(x, x) = -\frac{\partial p_x}{\partial u}(x, x) = -p'_x(x).$$

Together with (78) this yields the first equality in the statement of the lemma.

To obtain the second equality of the lemma, note that, since  $\deg p'_x \leq 2n-2$ , we may use  $\Sigma_x$  and (66) to obtain

$$\int_{-1}^1 p'_x(t)w(t)dt = \sum_{u \in S_x} \lambda_x(u)p'_x(u) = \lambda_x(-1)p'_x(-1) + \lambda(x)p'_x(x) + \lambda_x(1)p'_x(1).$$

## 6.2 Proof of Lemma 6.2

Fix a weight function  $w$  satisfying the conditions of Theorem 3.1 with  $\alpha = 0$ . In the following claims we examine more closely the behaviour of the polynomial  $p_x$  at the endpoints of the interval.

**Claim 6.6.** *There exists a constant  $C(w) > 0$  such that if  $x = \xi_r(a)$  then*

$$p_x(1) \leq C(w) \frac{\lambda(x)}{1-x}, \quad 1 \leq r \leq n-1 \text{ and } 0 \leq a < \infty, \quad (79)$$

$$p_x(-1) - 1 \leq C(w) \lambda(x) \min \left\{ \frac{1}{1+x}, n^2 \right\}, \quad 1 \leq r \leq n \text{ and } -\infty < a \leq 0. \quad (80)$$

*Proof.* By Lemma 5.1,

$$|P'_a(x)| \geq c(w) \sqrt{\frac{n}{\lambda(x)(1-x^2)}}, \quad x \in [\xi_1(0), \xi_n(0)].$$

Here, one may obtain some improvement to the following bounds when  $|a|$  is sufficiently large by using the full bound given by Lemma 5.1. However, these improvements do not seem to carry over to small values of  $|a|$ . In addition, by the upper bound in Corollary 3.2,

$$\begin{aligned} |P_a(1)| &= |\varphi(1)| \leq C(w) \sqrt{n}, \quad 0 \leq a < \infty, \\ |P_a(-1)| &= |\varphi(-1)| \leq C(w) \sqrt{n}, \quad -\infty < a \leq 0. \end{aligned}$$

Thus (79) follows using Lemma 6.5 and (44) since, for  $a > 0$ ,

$$p_x(1) = q_x(1) + \underline{p}_x(1) \leq q_x(1) = \frac{2}{x+1} \left( \frac{P_a(1)}{(1-x)P'_a(x)} \right)^2 \leq C(w) \frac{\lambda(x)}{1-x},$$

and similarly, if  $a = 0$ ,

$$p_x(1) \leq q_x(1) = \left( \frac{P_a(1)}{(1-x)P'_a(x)} \right)^2 \leq C(w) \frac{(1+x)\lambda(x)}{1-x}. \quad (81)$$

In a similar manner, we obtain (80) when  $2 \leq r \leq n$  or when  $r = 1$  and  $a = 0$  since

$$p_x(-1) - 1 \leq q_x(-1) = \left( \frac{2}{1-x} \right)^{\chi_{(-\infty, 0)}(a)} \left( \frac{P_a(-1)}{(1+x)P'_a(x)} \right)^2 \leq C(w) \frac{\lambda(x)}{1+x} \quad (82)$$

and since  $\frac{1}{1+x} \leq C(w)n^2$  by Proposition 3.6. It remains to prove (80) when  $r = 1$  and  $a < 0$ . For this case, by Lemma 5.1 and Proposition 3.6,

$$|P'_a(x)| \geq c(w) \sqrt{\frac{n}{\lambda(x)}} \max \left\{ \frac{|a|}{1-x}, \frac{1}{\sqrt{1-\xi_1(0)^2}} \right\} \geq c(w) \sqrt{\frac{n}{\lambda(x)}} \max \{|a|, n\}.$$

Thus, as in (82),

$$p_x(-1) - 1 \leq q_x(-1) = \left( \frac{2}{1-x} \right) \left( \frac{P_a(-1)}{(1+x)P'_a(x)} \right)^2 \leq C(w) \frac{\lambda(x)}{(1+x)^2} \min \left\{ \frac{1}{|a|^2}, \frac{1}{n^2} \right\}. \quad (83)$$

Fix  $\varepsilon(w) > 0$ , small enough for the following calculation. We consider separately two cases. First suppose that  $x \leq -1 + \frac{\varepsilon(w)}{n^2}$ . By (7) we have  $|a|(1+x) = |\varphi(x)/\psi(x)|$ . Hence, using (15) and (16), if  $\varepsilon(w)$  is sufficiently small then

$$|a|(1+x) \geq \frac{c(w)}{n}.$$

Plugging this into (83) proves (80) in this case. Now suppose that  $x \geq -1 + \frac{\varepsilon(w)}{n^2}$ . Here, (80) follows directly from (83).  $\square$

**Claim 6.7.** *There exists a constant  $C(w) > 0$  such that if  $x = \xi_r(a)$  for  $1 \leq r \leq n$  then*

$$-p'_x(1) \leq C(w)n^2 \frac{\lambda(x)}{1-x}, \quad -\infty < a < 0, \quad (84)$$

$$p'_x(-1) \leq C(w)n^2 \frac{\lambda(x)}{1+x}, \quad 0 < a < \infty. \quad (85)$$

*Proof.* By the upper bound in Corollary 3.2,

$$|P_a(\pm 1)| \leq C(w) (1 + |a|n) \sqrt{n}, \quad -\infty < a < \infty.$$

Thus, assuming that  $-\infty < a < 0$ , Lemma 6.5, (44) and Lemma 5.1 yield

$$\begin{aligned} -p'_x(1) &= -q'_x(1) - \underline{p}'_x(1) \leq -q'_x(1) = \frac{1}{1-x} \left( \frac{P_a(1)}{(1-x)P'_a(x)} \right)^2 \leq C(w) \frac{n(1+(-a)n)^2}{(1-x)^3 (P'_a(x))^2} \leq \\ &\leq C(w) \frac{(1+(-a)n)^2 \lambda(x)}{(1-x)^3} \min \left\{ \frac{(1-x)^2}{a^2}, 1 - \bar{x}^2 \right\} \leq \\ &\leq C(w) \frac{(1+(-a)n)^2 \lambda(x)}{(1-x)^2} \min \left\{ \frac{1-x}{a^2}, 1 + \bar{x} \right\} \leq \\ &\leq C(w) \frac{\lambda(x)}{(1-x)^2} \cdot \begin{cases} 1 + \bar{x} & |a| \leq \frac{1}{n} \\ a^2 n^2 (1 + \bar{x}) & \frac{1}{n} < |a| \leq \sqrt{\frac{1-x}{1+\bar{x}}} \\ n^2 (1-x) & |a| > \sqrt{\frac{1-x}{1+\bar{x}}} \end{cases}, \end{aligned} \quad (86)$$

where we recall the definition of  $\bar{x}$  from (42). The bound (84) now follows with the aid of Proposition 3.6.



In a similar manner, assuming that  $0 < a < \infty$ , we have

$$p'_x(-1) \leq q'_x(-1) = \frac{1}{1+x} \left( \frac{P_a(-1)}{(1+x)P'_a(x)} \right)^2 \leq C(w) \frac{(1+an)^2 \lambda(x)}{(1+x)^2} \min \left\{ \frac{1+x}{a^2}, 1-\bar{x} \right\},$$

yielding the bound (85).  $\square$

**Claim 6.8.** *There exists a constant  $C(w) > 0$  such that if  $x = \eta_r$  for  $1 \leq r \leq n-1$  then*

$$-p'_x(1) \leq C(w)n^2 \frac{\lambda(x)}{1-x}, \quad (87)$$

$$p'_x(-1) \leq C(w)n^2 \frac{\lambda(x)}{1+x}. \quad (88)$$

*Proof.* By the upper bound in Corollary 3.2,

$$|\psi(\pm 1)| \leq C(w)n\sqrt{n}.$$

Thus, (87) follows using Lemma 6.5, (44) and Lemma 5.1, by

$$\begin{aligned} -p'_x(1) &= -q'_x(1) - \underline{p}'_x(1) \leq -q'_x(1) = \frac{2}{1-x^2} \left( \frac{\psi(1)}{(1-x)\psi'(x)} \right)^2 \leq \\ &\leq C(w) \frac{n^3}{(1-x^2)(1-x)^2 (\psi'(x))^2} \leq C(w)n^2 \lambda(x) \frac{1+x}{1-x} \leq C(w)n^2 \frac{\lambda(x)}{1-x}. \end{aligned} \quad (89)$$

In a similar manner, (88) follows by

$$p'_x(-1) \leq q'_x(-1) = \frac{2}{1-x^2} \left( \frac{\psi(-1)}{(1+x)\psi'(x)} \right)^2 \leq C(w)n^2 \frac{\lambda(x)}{1+x}. \quad \square$$

*Proof of Lemma 6.2.* We first prove (68). We consider separately three cases.

1. Suppose  $x = \xi_r(a)$  for  $0 \leq a < \infty$ ,  $1 \leq r \leq n-1$ . In this case, the claim follows since  $p_x(1) \leq C(w) \frac{\lambda(x)}{1-x}$  by (79),  $\lambda_x(1) = 0$  since  $1 \notin S_x$  and  $\lambda_x(-1)p'_x(-1) \geq 0$  since  $\lambda_x(-1) = 0$  if  $a = 0$  and  $p'_x(-1) \geq 0$  if  $a > 0$  by Lemma 6.5.
2. Suppose  $x = \xi_r(a)$  for  $-\infty < a < 0$ ,  $1 \leq r \leq n$ . In this case, the claim follows since  $p_x(1) = 0$  by (65),  $\lambda_x(-1) = 0$  since  $-1 \notin S_x$ ,  $\lambda_x(1) \leq C(w) \cdot \frac{1}{n^2}$  by Lemma 3.7 and  $-p'_x(1) \leq C(w)n^2 \frac{\lambda(x)}{1-x}$  by (84).
3. Suppose  $x = \eta_r$  for  $1 \leq r \leq n-1$ . In this case, the claim follows since  $p_x(1) = 0$ ,  $p'_x(-1) \geq 0$ ,  $\lambda_x(1) \leq C(w) \cdot \frac{1}{n^2}$  by Lemma 3.7 and  $-p'_x(1) \leq C(w)n^2 \frac{\lambda(x)}{1-x}$  by (87).

We now prove (69). Again we consider separately three cases. We appeal to Proposition 3.6 to justify that  $\frac{1}{1+x} \leq C(w)n^2$  in the first and third cases.

1. Suppose  $x = \xi_r(a)$  for  $0 < a < \infty$ ,  $1 \leq r \leq n$ . In this case, the claim follows since  $p_x(-1) = 1$  by (65),  $\lambda_x(1) = 0$  since  $1 \notin S_x$ ,  $\lambda_x(-1) \leq C(w) \cdot \frac{1}{n^2}$  by Lemma 3.7 and  $p'_x(-1) \leq C(w)n^2 \frac{\lambda(x)}{1+x}$  by (85).
2. Suppose  $x = \xi_r(a)$  for  $-\infty < a \leq 0$ ,  $1 \leq r \leq n$ . In this case, the claim follows since  $p_x(-1) - 1 \leq C(w)\lambda(x) \min \left\{ \frac{1}{1+x}, n^2 \right\}$  by (80),  $\lambda_x(-1) = 0$  since  $-1 \notin S_x$  and  $\lambda_x(1)p'_x(1) \leq 0$  since  $\lambda_x(1) = 0$  if  $a = 0$  and  $p'_x(1) \leq 0$  if  $a < 0$  by Lemma 6.5.
3. Suppose  $x = \eta_r$  for  $1 \leq r \leq n-1$ . In this case, the claim follows since  $p_x(-1) = 1$ ,  $p'_x(1) \leq 0$ ,  $\lambda_x(-1) \leq C(w) \cdot \frac{1}{n^2}$  by Lemma 3.7 and  $p'_x(-1) \leq C(w)n^2 \frac{\lambda(x)}{1+x}$  by (88).  $\square$

### 6.3 Proof of Lemma 6.3

Integrating by parts, which is possible since  $w$  is absolutely continuous, we get

$$\begin{aligned}
\int_{-1}^1 p'_x(t)w(t)dt &= \left[ p_x(1)w(1) - p_x(-1)w(-1) \right] - \int_{-1}^1 p_x(t)w'(t)dt = \\
&= \left[ p_x(1)w(1) - p_x(-1)w(-1) \right] - \int_{-1}^1 \chi_{[-1,x]}(t)w'(t)dt - \int_{-1}^1 (p_x - \chi_{[-1,x]})(t)w'(t)dt = \\
&= \left[ p_x(1)w(1) - p_x(-1)w(-1) \right] - \left[ w(x) - w(-1) \right] - \int_{-1}^1 (p_x - \chi_{[-1,x]})(t)w'(t)dt = \\
&= -w(x) + \left[ p_x(1)w(1) - (p_x(-1) - 1)w(-1) \right] - \int_{-1}^1 (p_x - \chi_{[-1,x]})(t)w'(t)dt.
\end{aligned}$$

The lemma now follows since  $p_x(1) \geq 0$  and  $p_x(-1) \geq 1$  by Lemma 6.5, and

$$\begin{aligned}
\left| \int_{-1}^1 (p_x - \chi_{[-1,x]})(t)w'(t)dt \right| &\leq \int_{-1}^1 (p_x - \chi_{[-1,x]})(t)|w'(t)|dt \leq \\
&\leq \int_{-1}^1 (p_x - \underline{p}_x)(t)|w'(t)|dt = \int_{-1}^1 q_x(t)|w'(t)|dt
\end{aligned}$$

by another application of Lemma 6.5 and (73).

### 6.4 Proof of Lemma 6.4

If  $w$  satisfies the assumptions of Theorem 1.1 then using (45) we get

$$\int_{-1}^1 q_x(t)|w'(t)|dt \leq \frac{R}{m} \int_{-1}^1 q_x(t)w(t)dt = \frac{R}{m}\lambda(x).$$

If  $w$  satisfies the assumptions of Theorem 1.2 and  $w' \in L_p[-1, 1]$  for some  $p > 1$ , then using Holder's inequality, the first part of Proposition 5.5, and (45),

$$\begin{aligned} \int_{-1}^1 q_x(t) |w'(t)| dt &\leq \|q_x\|_{\frac{p}{p-1}} \|w'\|_p \leq \left( \max_{-1 \leq t \leq 1} q_x(t) \right)^{\frac{1}{p}} \left( \int_{-1}^1 q_x(t) dt \right)^{1-\frac{1}{p}} \|w'\|_p \leq \\ &\leq C(w) \left( \frac{1}{m} \int_{-1}^1 q_x(t) w(t) dt \right)^{1-\frac{1}{p}} \|w'\|_p = C(w) \left( \frac{\lambda(x)}{m} \right)^{1-\frac{1}{p}} \|w'\|_p \leq C(w) \lambda(x)^{1-\frac{1}{p}}. \end{aligned}$$

## 7 Discontinuous weights

In this section we prove Theorem 1.3. The theorem follows as an immediate consequence, using Lemma 3.8, from the following proposition.

**Proposition 7.1.** *Suppose  $w$  is a weight function on  $[-1, 1]$  satisfying the assumptions of Theorem 1.3. For every  $\varepsilon > 0$  there exists an  $n_0 = n_0(w, \varepsilon)$  such that if  $n \geq n_0$  then for every differentiability point  $x \in (-1, 1)$  of  $\pi$ ,*

$$\begin{aligned} -C(w) \frac{\lambda(x)}{1-x} - \varepsilon - C(w) \sum_{i=1}^L \min \left\{ \frac{1}{n^2(s_i - x)^2}, 1 \right\} &\leq \pi'(x) - w(x) \leq \\ &\leq C(w) \lambda(x) \min \left\{ \frac{1}{1+x}, n^2 \right\} + \varepsilon + C(w) \sum_{i=1}^L \min \left\{ \frac{1}{n^2(s_i - x)^2}, 1 \right\}. \end{aligned} \quad (90)$$

Figures 8, 9 and 10 show the graphs of  $\pi$ ,  $\pi' - w$  and  $\lambda$  for a discontinuous weight function  $w$  satisfying the assumptions of Theorem 1.3.

The proof of Proposition 7.1 follows the same strategy as that of Theorem 1.2. Indeed, all the ingredients used in the proof of Theorem 1.2, with the exceptions of Lemma 6.3 and Lemma 6.4, are proved for weight functions satisfying the assumptions of Theorem 3.1 with  $\alpha = 0$  and are thus valid also for weight functions satisfying the assumptions of Theorem 1.3. We prove only the lower bound in Proposition 7.1 as the proof of the upper bound is similar. A replacement for Lemmas 6.3 and 6.4 is provided by the next two lemmas.

We recall that by the assumptions of Theorem 1.3,  $w$  has left and right limits at each point  $s$ , which will be denoted  $w(s-)$  and  $w(s+)$ , respectively. We remind the reader of the definition of  $q_x$  from Section 5 and the definition of  $p_x$  from Section 6.

**Lemma 7.2.** *Suppose  $w$  satisfies the assumptions of Theorem 1.3. Then for every  $-1 < x < 1$ ,  $x \notin \{s_1, \dots, s_L\}$ ,*

$$\int_{-1}^1 p'_x(t) w(t) dt \leq -w(x) + p_x(1)w(1) - \sum_{i=1}^L (w(s_i+) - w(s_i-)) (p_x - \chi_{[-1, x]})(s_i) + \int_{-1}^1 q_x(t) |w'(t)| dt. \quad (91)$$

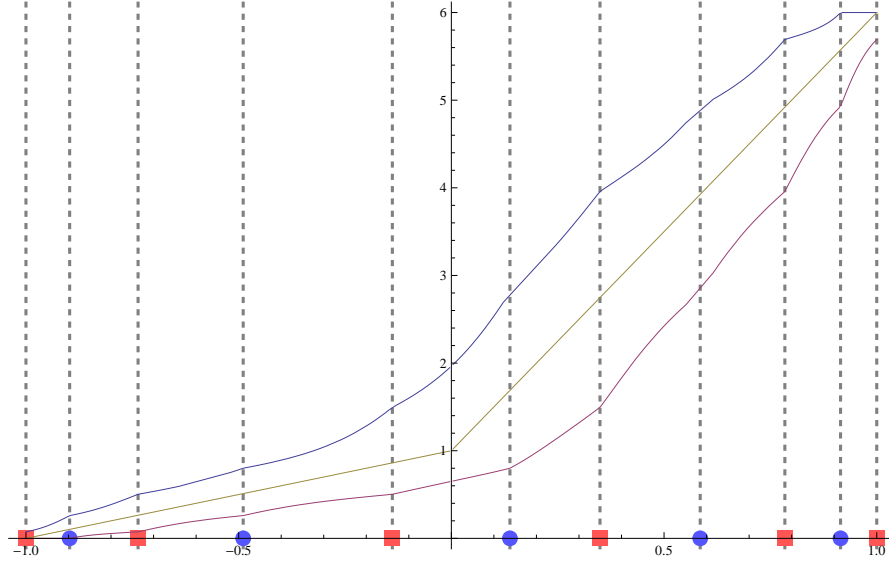


Figure 8: A plot of  $\pi$  (top graph),  $\int_{-1}^x w(t)dt$  (middle graph) and  $\underline{\pi}$  (bottom graph) for  $n = 5$  and the weight function  $w$  defined by  $w(t) = 1$  if  $t < 0$  and  $w(t) = 5$  if  $t \geq 0$ . Observe that  $\pi$  lies above the graph of the integral, as the Chebyshev-Markov-Stieltjes inequalities guarantee (see (2)). The circles on the axis denote the nodes of the Gaussian quadrature and the squares denote the nodes of the Lobatto quadrature.

*Proof.* Using integration by parts on the interval  $[-1, s_1]$ , which is possible since  $w$  is absolutely continuous on  $[-1, s_1]$  when interpreting  $w(s_1)$  as  $w(s_1-)$ , we have

$$\begin{aligned} \int_{-1}^{s_1} p'_x(t)w(t)dt &= \left[ p_x(s_1)w(s_1-) - p_x(-1)w(-1) \right] - \int_{-1}^{s_1} p_x(t)w'(t)dt = \\ &= \left[ p_x(s_1)w(s_1-) - p_x(-1)w(-1) \right] - \int_{-1}^{s_1} \chi_{[-1,x]}(t)w'(t)dt - \int_{-1}^{s_1} \left( p_x - \chi_{[-1,x]} \right)(t)w'(t)dt. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{s_L}^1 p'_x(t)w(t)dt &= \left[ p_x(1)w(1) - p_x(s_L)w(s_L+) \right] - \int_{s_L}^1 p_x(t)w'(t)dt = \\ &= \left[ p_x(1)w(1) - p_x(s_L)w(s_L+) \right] - \int_{s_L}^1 \chi_{[-1,x]}(t)w'(t)dt - \int_{s_L}^1 \left( p_x - \chi_{[-1,x]} \right)(t)w'(t)dt \end{aligned}$$

and, for every  $2 \leq i \leq L$ ,

$$\begin{aligned} \int_{s_{i-1}}^{s_i} p'_x(t)w(t)dt &= \left[ p_x(s_i)w(s_i-) - p_x(s_{i-1})w(s_{i-1}+) \right] - \int_{s_{i-1}}^{s_i} p_x(t)w'(t)dt = \\ &= \left[ p_x(s_i)w(s_i-) - p_x(s_{i-1})w(s_{i-1}+) \right] - \int_{s_{i-1}}^{s_i} \chi_{[-1,x]}(t)w'(t)dt - \int_{s_{i-1}}^{s_i} \left( p_x - \chi_{[-1,x]} \right)(t)w'(t)dt. \end{aligned}$$

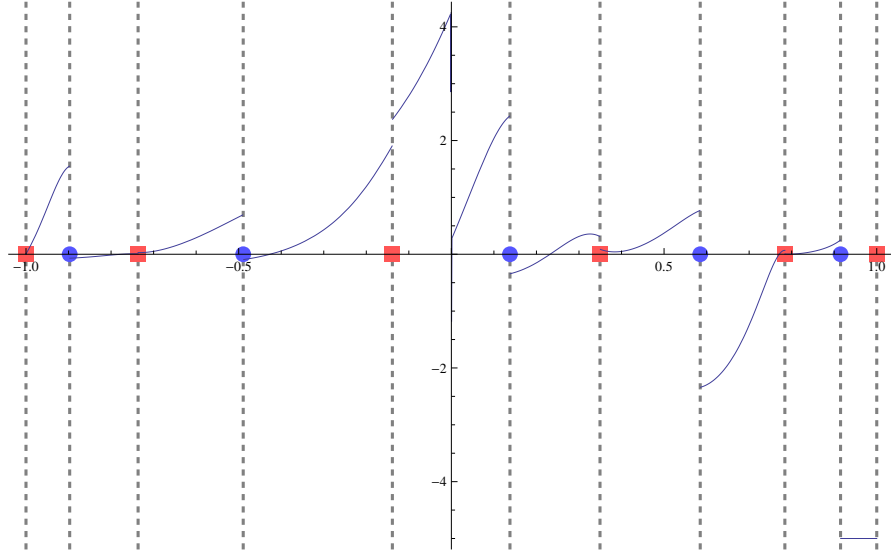


Figure 9: A plot of the function  $\pi' - w$  for  $n = 5$  and the weight function  $w$  defined by  $w(t) = 1$  if  $t < 0$  and  $w(t) = 5$  if  $t \geq 0$ . The circles on the axis denote the nodes of the Gaussian quadrature and the squares denote the nodes of the Lobatto quadrature. We note that the jump at  $t = 0$  is exactly the jump of  $w$  at this point as we know that  $\pi$  is analytic there, see Section 2.

Summing these  $L + 1$  equalities and noticing that

$$\int_{-1}^1 \chi_{[-1,x]}(t) w'(t) dt = w(x) - w(-1) - \sum_{i=1}^L (w(s_i+) - w(s_i-)) \chi_{[-1,x]}(s_i)$$

we get

$$\begin{aligned} \int_{-1}^1 p'_x(t) w(t) dt &= -w(x) + (1 - p_x(-1)) w(-1) + p_x(1) w(1) - \\ &\quad - \sum_{i=1}^L (w(s_i+) - w(s_i-)) (p_x - \chi_{[-1,x]})(s_i) - \int_{-1}^1 (p_x - \chi_{[-1,x]})(t) w'(t) dt. \end{aligned}$$

The lemma now follows since  $p_x(-1) \geq 1$  by Lemma 6.5, and

$$\begin{aligned} \left| \int_{-1}^1 (p_x - \chi_{[-1,x]})(t) w'(t) dt \right| &\leq \int_{-1}^1 (p_x - \chi_{[-1,x]})(t) |w'(t)| dt \leq \\ &\leq \int_{-1}^1 (p_x - \underline{p}_x)(t) |w'(t)| dt = \int_{-1}^1 q_x(t) |w'(t)| dt \end{aligned}$$

by another application of Lemma 6.5 and (73). □

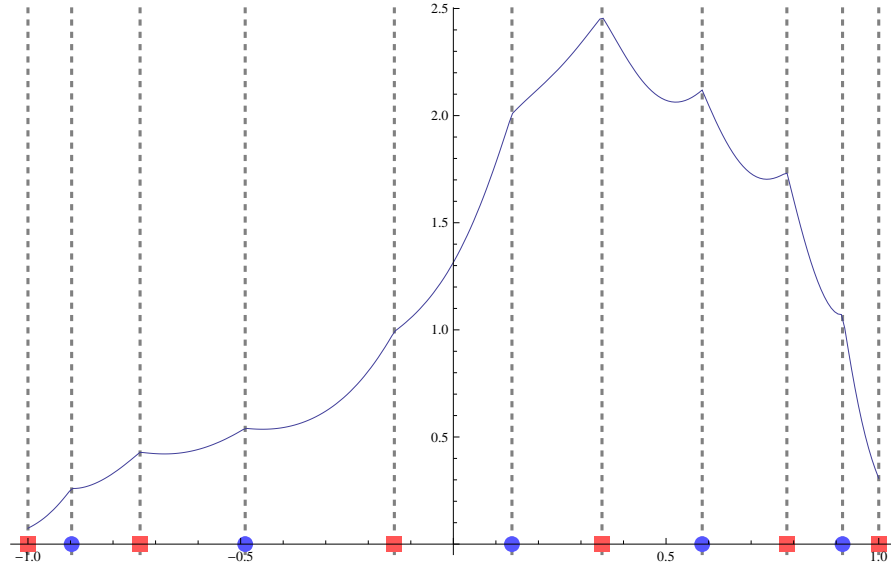


Figure 10: A plot of the function  $\lambda$  for  $n = 5$  and the weight function  $w$  defined by  $w(t) = 1$  if  $t < 0$  and  $w(t) = 5$  if  $t \geq 0$ . The circles on the axis denote the nodes of the Gaussian quadrature and the squares denote the nodes of the Lobatto quadrature.

**Lemma 7.3.** *Suppose  $w$  satisfies the assumptions of Theorem 1.3. For every  $\varepsilon > 0$  there exists an  $n_0 = n_0(w, \varepsilon)$  such that if  $n \geq n_0$  then for every  $-1 < x < 1$ ,*

$$\int_{-1}^1 q_x(t) |w'(t)| dt \leq \varepsilon.$$

*Proof.* Let  $\tilde{\varepsilon} > 0$ . If  $w$  satisfies the assumptions of Theorem 1.3 then there is a  $\delta = \delta(w, \tilde{\varepsilon}) > 0$  such that  $\int_I |w'(t)| dt < \tilde{\varepsilon}$  for every interval  $I \subseteq [-1, 1]$  such that  $|I| \leq 2\delta$ . Thus using both statements of Proposition 5.5,

$$\begin{aligned} \int_{-1}^1 q_x(t) |w'(t)| dt &= \int_{\{-1 \leq t \leq 1: |t-x| \leq \delta\}} q_x(t) |w'(t)| dt + \int_{\{-1 \leq t \leq 1: |t-x| > \delta\}} q_x(t) |w'(t)| dt \leq \\ &\leq C(w) \int_{\{-1 \leq t \leq 1: |t-x| \leq \delta\}} |w'(t)| dt + \frac{C(w)}{n\delta^2} \int_{\{-1 \leq t \leq 1: |t-x| > \delta\}} |w'(t)| dt \leq C(w)\tilde{\varepsilon} + \frac{C(w)}{n\delta^2} \leq C(w)\tilde{\varepsilon} \end{aligned}$$

when  $n \geq \frac{1}{\tilde{\varepsilon}\delta(w, \tilde{\varepsilon})^2}$ . The stated bound follows by choosing  $\tilde{\varepsilon} = c(w)\varepsilon$ .  $\square$

We now prove Proposition 7.1.

*Proof of the lower bound in Proposition 7.1.* When  $x \geq \xi_n(0)$  the bound follows by taking  $C(w)$  large enough, using the fact that  $\pi$  is non-decreasing by (1) so that  $\pi'(x) \geq 0$ , using Corollary 5.4 and using Proposition 3.6 to see that  $\frac{\lambda(x)}{1-x} \geq c(w)$ .

For  $x < \xi_n(0)$ , combining Lemmas 6.1, 6.2 and 7.2 yields

$$\pi'(x) \geq w(x) - C(w) \frac{\lambda(x)}{1-x} + \sum_{i=1}^L (w(s_i+) - w(s_i-)) (p_x - \chi_{[-1,x]})(s_i) - \int_{-1}^1 q_x(t) \cdot |w'(t)| dt.$$

By Lemma 6.5, (73) and Proposition 5.5, for each  $i$ ,

$$\begin{aligned} |w(s_i+) - w(s_i-)| (p_x - \chi_{[-1,x]})(s_i) &\leq C(w) q_x(s_i) \leq \\ &\leq C(w) \min \left\{ \frac{1}{n^2 \sqrt{1-s_i^2} (s_i - x)^2}, 1 \right\} \leq C(w) \min \left\{ \frac{1}{n^2 (s_i - x)^2}, 1 \right\}. \end{aligned}$$

The proposition follows using Lemma 7.3.  $\square$

**Remark 7.4.** *As can be seen from the proof, for every  $1 \leq i \leq L$ , the term  $\min \left\{ \frac{1}{n^2 (s_i - x)^2}, 1 \right\}$  may be omitted, either from the lower bound, if  $w(s_i+) \geq w(s_i-)$ , or from the upper bound, if  $w(s_i+) \leq w(s_i-)$ .*

## 8 Discussion and open problems

The main result in our work is a differential version of the Chebyshev-Markov-Stieltjes inequalities given by Theorem 1.1, Theorem 1.2 and Theorem 1.3. The Chebyshev-Markov-Stieltjes inequalities hold for every weight function (and more generally, any measure). In what generality does a differential version of the inequalities hold? Our results show that some version holds for all weight functions which are absolutely continuous and bounded away from zero, and also for a certain class of discontinuous weight functions. To what extent are such assumptions on the weight function necessary for the result? Do similar results hold for Jacobi weight functions, when  $w(x) = (1-x)^\alpha(1+x)^\beta$ ?

In addition, what are the best possible error terms in Theorem 1.1? Writing  $x = \xi_i(a)$ , these error terms may be improved for certain ranges of  $i$  and  $a$ , see Section 6.2, e.g., in the proof of Claim 6.6 and in estimates (81), (86) and (89), but it is not clear what would be the form of the sharp bounds. This question may be asked also for weight functions satisfying the assumptions of Theorem 1.2 or Theorem 1.3.

To make the bounds in our theorems fully effective one would require explicit bounds for the constants  $C(w)$  appearing in them. When  $w$  satisfies the conditions of Theorem 1.1 quantitative estimates for  $C(w)$  in terms of the Lipschitz constant and minimal value of  $w$  may be obtained from our proof (including the proof of Theorem 3.1 for this case in Appendix A, where one may obtain quantitative estimates for  $\ell_n$  and  $f_n$  via the Korovskii comparison principle and [10, (11.3.6)]). We do not know to similarly bound the constants appearing in Theorem 1.2 and Theorem 1.3 by parameters depending only on the minimal value and the regularity of  $w$ . This is due to the fact that the dependence on  $w$  in the constants appearing in Theorem 3.1 is non-explicit.

## Acknowledgements

We thank Vladimir Badkov, Percy Deift, Eli Levin, Doron Lubinsky, Paul Nevai and Mikhail Sodin for helpful remarks and discussions during the course of this work.

## References

- [1] V. M. Badkov, Asymptotic and extremal properties of orthogonal polynomials in the presence of singularities in the weight, *Trudy Mat. Inst. Steklov.* **198** (1992), 41–88 (in Russian); translation in *Proc. Steklov Inst. Math.* **1994**, no. 1 (198), 37–82.
- [2] V. M. Badkov, Approximation of functions in a uniform metric by Fourier sums in orthogonal polynomials, *Trudy Mat. Inst. Steklov.* **145** (1980), 20–62, 249 (in Russian); translation in *Proc. Steklov Inst. Math.* **1981**, no. 1 (145), 19–65.
- [3] P. Borwein and T. Erdélyi, *Polynomials and polynomial inequalities*, Graduate Texts in Mathematics, 161, Springer, New York, 1995.
- [4] P. Erdős and B. A. Lengyel, On fundamental functions of Lagrangean interpolation, *Bull. Amer. Math. Soc.* **44** (1938), no. 12, 828–834.
- [5] P. Erdős and P. Turán, On interpolation. II. On the distribution of the fundamental points of Lagrange and Hermite interpolation, *Ann. of Math. (2)* **39** (1938), no. 4, 703–724.
- [6] Ya. L. Geronimus, *Orthogonal polynomials: Estimates, asymptotic formulas, and series of polynomials orthogonal on the unit circle and on an interval*, Authorized translation from the Russian, Consultants Bureau, New York, 1961.
- [7] S. Karlin and W. J. Studden, *Tchebycheff systems: With applications in analysis and statistics*, Pure and Applied Mathematics, Vol. XV Interscience Publishers John Wiley & Sons, New York, 1966.
- [8] A. Kuijlaars, The minimal number of nodes in Chebyshev type quadrature formulas, *Indag. Math. (N.S.)* **4** (1993), no. 3, 339–362.
- [9] A. Kuijlaars, Chebyshev-type quadrature for Jacobi weight functions, *J. Comput. Appl. Math.* **57** (1995), no. 1-2, 171–180.
- [10] G. Szegő, *Orthogonal polynomials*, fourth edition, Amer. Math. Soc., Providence, RI, 1975.



## A Appendix: Remarks on Badkov's theorem

In this section we provide some remarks on Badkov's Theorem 3.1 including a proof for the case that the function  $h$  is Lipschitz continuous and  $\alpha = 0$ , the main case in our proof of Theorem 1.1.

Write  $(\varphi_n)$  and  $(\psi_n)$  for the orthogonal polynomials on  $[-1, 1]$  with respect to the weight functions  $w(t)$  and  $(1 - t^2)w(t)$ , respectively, so that in the notation of our paper,  $\varphi = \varphi_n$  and  $\psi = \psi_{n-1}$ . The starting point for the theorem is a relation between  $(\varphi_n), (\psi_n)$  and orthogonal polynomials on the unit circle for a related weight function. Define  $\tilde{w} : [-\pi, \pi] \rightarrow [0, \infty)$  by

$$\tilde{w}(\theta) := w(\cos \theta) |\sin \theta|. \quad (92)$$

Let  $(\phi_n)$  be the orthogonal polynomials on the unit circle with respect to  $\tilde{w}$ , that is,  $\deg(\phi_n) = n$  and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_n(e^{i\theta}) \phi_m(e^{i\theta}) \tilde{w}(\theta) d\theta = \delta_{n,m},$$

normalized to have real positive leading coefficients, which we denote by  $\ell_n$ . We also let  $f_n := \phi_n(0)$  be the constant term of  $\phi_n$ . The following relation, a consequence of [10, Theorem 11.5], connects the three systems of orthogonal polynomials,

$$\sqrt{\frac{2}{\pi}} e^{-in\theta} \phi_{2n}(e^{i\theta}) = \sqrt{1 + \frac{f_{2n}}{\ell_{2n}}} \varphi_n(\cos \theta) + i \sqrt{1 - \frac{f_{2n}}{\ell_{2n}}} \sin \theta \psi_{n-1}(\cos \theta), \quad -\pi \leq \theta \leq \pi, n \geq 1. \quad (93)$$

The next lemma uses this relation to show that Theorem 3.1 is equivalent to estimating  $|\phi_{2n}|$  on the unit circle.

**Lemma A.1.** *If*

$$\int_{-\pi}^{\pi} |\log \tilde{w}(\theta)| d\theta < \infty \quad (94)$$

*then there exist constants  $C(w), c(w) > 0$  such that for every  $n \geq 1$ ,*

$$c(w) |\phi_{2n}(e^{i\theta})| \leq |\varphi_n(x)| + \sqrt{1 - x^2} |\psi_{n-1}(x)| \leq C(w) |\phi_{2n}(e^{i\theta})|, \quad x = \cos \theta, \quad -\pi \leq \theta \leq \pi.$$

*Proof.* Equation 11.3.12 in [10] implies that  $\ell_n$  tends to a positive limit, whence equation 11.3.6 in [10] implies that  $f_n$  tends to zero. The lemma follows from these facts using equation (93).  $\square$

Theorems 1.2 and 1.4 of Badkov [1] give two-sided estimates on  $|\phi_n|$  on the unit circle under rather general assumptions on  $\tilde{w}$  which include the assumptions of Theorem 3.1. When the weight  $w$  is assumed to be Lipschitz continuous such estimates may also be derived by means of the Korovkin comparison theorem. We proceed to describe this derivation for the case  $\alpha = 0$  which we are interested in (see also [2]) and comment briefly on the more general case at the end of the appendix.

Assume now that  $w$  satisfies the assumptions of Theorem 1.1. The proof will follow by a comparison argument. Denote by  $u$  the constant weight function,  $u \equiv 1$  on  $[-1, 1]$ . Let  $(L_n)$  be the Legendre

polynomials, defined in (27), orthogonal with respect to  $u$ . The polynomials  $(L'_n)$  are orthogonal with respect to the weight  $1 - t^2$  and we have the normalizations [10, (4.21.7), (4.3.3)]

$$\int_{-1}^1 L_n^2(t) dt = \frac{1}{n + \frac{1}{2}} \quad \text{and} \quad \int_{-1}^1 L_n'^2(t)(1 - t^2) dt = \frac{n(n+1)}{n + \frac{1}{2}}.$$

Define also the orthonormal versions,

$$\overline{L}_n = \sqrt{n + \frac{1}{2}} L_n \quad \text{and} \quad \overline{L}'_n = \sqrt{\frac{n + \frac{1}{2}}{n(n+1)}} L'_n.$$

**Lemma A.2.** *There exist absolute constants  $C, c > 0$  such that for every  $-1 < x < 1$  and  $n \geq 1$ ,*

$$c \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{1/2} \leq |\overline{L}_n(x)| + \sqrt{1-x^2} |\overline{L}'_n(x)| \leq C \min \left\{ n, \frac{1}{\sqrt{1-x^2}} \right\}^{1/2}. \quad (95)$$

*Proof.* The upper bound follows by standard estimates of Jacobi polynomials [10, Theorem 7.32.2].

For the lower bound, define  $F_n(x) := n(n+1)L_n^2(x) + (1-x^2)L_n'^2(x)$ . It suffices to show that

$$F_n(x) \geq c n \min \left\{ \frac{1}{\sqrt{1-x^2}}, n \right\}. \quad (96)$$

This estimate holds by [10, Theorem 8.21.13] when  $n \geq n_0$  and  $1-x^2 \geq \frac{c'}{n^2}$ , for some  $n_0, c' > 0$ . It holds trivially for  $n < n_0$ , adjusting the constant  $c$  as necessary, since  $L_n$  has no double root. Finally, (96) follows also when  $n \geq n_0$  and  $1-x^2 \leq \frac{c'}{n^2}$  by observing that the differential equation for the Legendre polynomials implies that  $F_n$  is monotone decreasing on  $[-1, 0]$  and monotone increasing on  $[0, 1]$ , see [10, (7.3.4)].  $\square$

Since  $w$  satisfies the assumptions of Theorem 1.1 we may apply the Korovs comparison theorem [10, Theorem 7.1.3] to obtain

$$|\varphi_n(x)| \leq C(w) (|\overline{L}_{n-1}(x)| + |\overline{L}_n(x)|), \quad |\psi_{n-1}(x)| \leq C(w) (|\overline{L}'_{n-1}(x)| + |\overline{L}'_n(x)|), \quad (97)$$

$$|\overline{L}_n(x)| \leq C(w) (|\varphi_{n-1}(x)| + |\varphi_n(x)|), \quad |\overline{L}'_n(x)| \leq C(w) (|\psi_{n-2}(x)| + |\psi_{n-1}(x)|). \quad (98)$$

The upper bound in (13) (for  $\alpha = 0$ ) now follows by combining the inequalities in (97) and using Lemma A.2. To obtain the lower bound note first that by [10, (11.4.6)] we have

$$|\phi_n(z)| = \frac{|\ell_{n+1}\phi_{n+1}(z) - f_{n+1}z^{n+1}\overline{\phi_{n+1}(z)}|}{|\ell_n z|} \leq C(w) |\phi_{n+1}(z)|, \quad |z| = 1, n \geq 0, \quad (99)$$

where we have used that  $\ell_n \rightarrow \ell > 0$  and  $f_n \rightarrow 0$  as in the proof of Lemma A.1. Finally, the lower bound follows by combining the inequalities in (98), using Lemma A.2, applying Lemma A.1 and using (99) twice.

We finish by briefly remarking on the method used in Badkov's paper [1] from which the general case of Theorem 3.1 follows. Badkov begins by upper bounding  $|\phi_n|$  on the unit circle via the so-called

Szegő function  $\pi$  associated with the weight  $\tilde{w}$ . This bound is up to an error  $(1 + \delta_n \sqrt{n})$  for a related quantity  $\delta_n$  [1, Lemma 4.2] (see also [6, Theorem 3.6 and Theorem 4.10]). The advantage of such a bound is that both the Szegő function and the quantity  $\delta_n$  are multiplicative in the weight function  $\tilde{w}$  [1, Lemma 4.1], thus allowing one to bound them separately for the factors  $|\sin \theta|$  and  $w(\cos \theta)$  present in (92). This task is undertaken in [1, Theorem 2.1 and Theorem 4.1]. Condition (12) is used for estimating  $\delta_n$  for the factor  $w(\cos \theta)$  via results in [6, Section 3.7]. This provides the required upper bound on  $|\phi_n|$ . To obtain the lower bound, the function  $|\phi'_n \phi_n|$  is estimated from below by the Christoffel-Darboux kernel [1, Lemma 11.1]. This kernel is then estimated from below [1, Theorem 9.2] and an upper bound for  $|\phi'_n|$  is derived from the upper bound for  $|\phi_n|$ .